Question

A viscous incompressilbe fluid with constant density ρ and constant dynamic viscosity μ flows unsteadily in the (x, y)-plane. There are no body foces. Show that when the Reynolds number Re is much less than one the flow is governed by the SLOW FLOW EQUATIONS

$$\begin{aligned} \nabla p &= \mu \nabla^2 \underline{q} \\ \text{div}\underline{q} &= 0 \end{aligned}$$

where p and $\underline{q}$ denote respectively the pressure and velocity of the flow.
Show further that, for two-dimensional flow, if a stream function $\phi(x, y)$ is defined in the normal way, then ϕ satisfies the biharmonic equation

$$\nabla^4 \phi = 0$$

In terms of plane polar coordinates (r, θ), a wedge of i ncreasing angle is formed by hinging two infinte plates $\theta = \pm\Omega t$ at $r = 0$. The plates thus move with angular velocities $\pm\Omega$. The value of Ω is chosen so that the plates move slowly apart and slow viscous flow takes place between them. The velocity of the fluid is denoted by $\underline{q} = u\underline{e}_r + v\underline{e}_\theta$ where $\underline{e}_r$ and $\underline{e}_\theta$ are unit vectors in the r and θ directions respectively.
Given that the stream function $\phi(r, \theta)$ may be defined by

$$\begin{aligned} u &= \frac{1}{r}\phi_\theta \\ v &= -\phi_r \end{aligned}$$

and that, in spherical polar coordinates

$$\nabla^2 \phi = \phi_{rr} + \frac{1}{r}\phi_r + \frac{1}{r^2}\phi_{\theta\theta}$$

verify that

$$\phi = r^2 \left[A_1(t) \sin 2\theta + A_2(t)\theta \right]$$

is a suitable stream function for the flow, where $A_1(t)$ and $A_2(t)$ are functions that should be determined. Show further that the mass flow passing across any arc $r = a$ is independent of time.

Answer

Start with unsteady Navier-Stokes.

$$\left.\begin{aligned} \underline{q}_t + (\underline{q}.\nabla)\underline{q} &= \frac{-1}{\rho}\nabla p + \nu\nabla^2\underline{q} \\ div(\underline{q}) &= 0 \end{aligned}\right\}$$

Non-dimensionalize by setting $\underline{x} = L\overline{\underline{x}}$, $\underline{q} = U\overline{\underline{q}}$, $p = \left(\frac{\mu U}{L}\right)\overline{p}$, where L and U are a representitive length and speed in the flow. Also set $t = \left(\frac{L}{U}\right)\overline{t}$.

$$\Rightarrow \quad \left.\begin{aligned} \frac{U^2}{L}(\overline{\underline{q}}_{\overline{t}} + (\overline{\underline{q}}.\overline{\nabla})\overline{\underline{q}}) &= \frac{-\mu U}{\rho L^2}\overline{\nabla}\overline{p} + \frac{\nu U}{L^2}\overline{\nabla}^2\overline{\underline{q}} \\ \overline{\nabla}.\overline{\underline{q}} &= 0 \end{aligned}\right\}$$

The momentum equation now becomes $Re[\overline{\underline{q}}_{\overline{t}} + (\overline{\underline{q}}.\overline{\nabla})\overline{\underline{q}}] = -\overline{\nabla}\overline{p} + \overline{\nabla}^2\overline{\underline{q}}$

So for $Re \ll 1$ we have to leading order (re-dimensionalize)

$$\left.\begin{aligned} &\nabla p = \mu\nabla^2\underline{q} \\ &div(\underline{q}) = 0 \end{aligned}\right\}$$

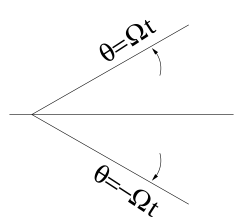
Now if we define $u = \psi_y$, $v = -\psi_x$ then $\div(\underline{q})$.

Also since $curl(grad(p)) \equiv 0$, we have $\nabla^2 curl(\underline{q})$.

Now, $curl(\underline{q}) = \begin{vmatrix} \underline{i} & \underline{j} & \underline{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ \psi_x & -\psi_y & 0 \end{vmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\psi_{xx} - \psi_{yy} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ -\nabla^2\psi \end{pmatrix}$

$\Rightarrow \nabla^2(-\nabla^2\psi) = 0$, so ψ satisfies the biharmonic equation

$$\nabla^4\psi = 0$$



Now $\psi = \mathrm{r}^2[\mathrm{A}_1 \sin 2\theta + \mathrm{A}_2\theta]$

$$\begin{aligned} \nabla^2\psi &= 2[A_1 \sin 2\theta + A_2\theta] + [A_1 \sin 2\theta + A_2\theta] + [-4A_1 \sin 2\theta] \\ &= 4A_2\theta \end{aligned}$$

But now $\nabla^2(4A_2\theta 0 = 0$.

So certainly the given ψ satisfies the biharmonic equation.

Boundary conditions, (symmetric so need only look at $\theta = \Omega t$)

At $\theta = \Omega t$ the plate velocity is $0\underline{\tilde{e}}_r + \Omega\underline{\tilde{e}}_\theta r$

$\Rightarrow$ we need

$$u = 0, \quad v = r\Omega \quad \text{at } \theta = \Omega \mathrm{t}$$

Thus $\psi_\theta = 0$, $\psi_r = -r\Omega$ at $\theta = \Omega t$
$\Rightarrow$

$$\begin{aligned} r^2[A_1(t)2\cos 2\Omega t + A_2(t)] &= 0 \\ 2r[A_1(t)\sin 2\Omega t + A_2\Omega t] &= -r\Omega \end{aligned}$$

Solving these $\Rightarrow$

$$\begin{aligned} A_1(t) &= \frac{-\Omega}{2[\sin 2\Omega t - 2\Omega t\cos 2\Omega t]} \\ A_2(t) &= \frac{\Omega\cos 2\Omega t}{[\sin 2\Omega t - 2\Omega t\cos 2\Omega t]} \end{aligned}$$

$$\psi = \frac{r^2}{(\sin 2\Omega t - 2\Omega t\cos 2\Omega t)}\left[\frac{-\sin 2\theta}{2} + \theta\cos 2\Omega t\right]$$

Mass flow

$$\begin{aligned} \rho\int_{\theta=\Omega t}^{\Omega t} ur\,d\theta &= \rho\int_{-\Omega t}^{\Omega t}\psi_\theta\,d\theta \\ &= \rho[\psi(r_1,\Omega t) - \psi(r_1,-\Omega t)] \end{aligned}$$

$$\begin{aligned} &= \frac{\rho r^2\Omega}{(\sin 2\Omega t - 2\Omega t\cos 2\Omega t)}\left[\frac{-\sin 2\Omega t}{2} + \Omega t\cos 2\Omega t\right. \\ &\quad \left. -\frac{\sin 2\Omega t}{2} + \Omega t\cos 2\Omega 4\right] \\ &= -\rho r^2\Omega \end{aligned}$$

$\Rightarrow$ independent of t.