Question

A solid sphere $r = a$ is held fixed in an otherwise uniform free stream of speed U which flows parallel to the x-axis. The REynolds number of the stream is small. The fluid has constant density ρ and constant kinematic viscosity ν and its velocity it denoted by $\underline{q} = u\underline{e}_r + v\underline{e}_\theta$ where $\underline{e}_r$ and $\underline{e}_\theta$ are unit vectors in the r and θ directions respectively and (t, θ, ϕ) are spherical polar coordinates. The flow is axisymmetric (i.e. independent of ϕ).
YOU MAY ASSUME that the Stoke stream function $\psi(r, \theta)$ for the flow is defined by

$$\begin{aligned} u &= \frac{1}{r^2 \sin\theta} \frac{\partial \psi}{\partial \theta} \\ v &= -\frac{1}{r \sin\theta} \frac{\partial \psi}{\partial r} \end{aligned}$$

and that the slow flow equations in spherical polar coordinates are

$$\left(\frac{\partial^2}{\partial^2 r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} - \frac{\cot\theta}{r^2} \frac{\partial}{\partial \theta} \right)^2 \psi = 0.$$

(i) Determine the boundary conditions for the flow.

(ii) By assuming a stream function of the form

$$\psi = U g(r) \sin^2\theta$$

obtain u and v

(iii) Sketch the flow streamlines.

(iv) By considering the orders of magnitude of $\underline{q}$ for large calues of a suitably non-dimensionalised r, show that slow flow theory (and hence the analysis carried out above) is invalid if $r \geq O(1/Re)$.

Answer

(i) at $r = a$ need $\psi = \psi_r = 0$ (no-slip).

As $r \to \infty$ we have $\underline{q} = U\hat{e}_x$.

Now $\hat{e}_r = \cos\theta \hat{e}_x + \sin\theta \hat{e}_y, \quad \hat{e}_\theta = -\sin\theta \hat{e}_x + \cos\theta \hat{e}_y$

$\Rightarrow \hat{e}_r - \sin\theta \hat{e}_\theta$

$$\frac{1}{r^2 \sin\theta}\psi_\theta = U\cos\theta, \quad \frac{-1}{r\sin\theta}\psi_r = -U\sin\theta$$

$\Rightarrow \psi_r \sim \frac{1}{2}Ur^2\sin^2\theta$ as $r \to \infty$

Now try $\psi = Ug(r)\sin^2\theta$

(ii) $\left(\frac{\partial^2}{\partial r^2} + \frac{1}{v^2}\frac{\partial^2}{\partial \theta^2} - \frac{\cot\theta}{r^2}\frac{\partial}{\partial\theta}\right) Ug(r)\sin^2\theta$

$= U\left[g'' sin^2\theta + \frac{g}{r^2}(2\cos^2\theta - 2\sin^2\theta) - \frac{\cos\theta}{r^2\sin\theta} g 2\sin\theta\cos\theta\right]$

$= U\left[g''\sin^2\theta - \frac{2g}{r^2}\sin^2\theta\right] = U\sin^2\theta\left[g'' - \frac{2g}{r^2}\right]$

Applying the operator again$\Rightarrow$

$$\begin{aligned}
& U\sin^2\theta\left(g'' - \frac{2g}{r^2}\right)'' + \frac{U}{r^2}\left(g'' - \frac{2g}{r^2}\right)(2\cos^2\theta - 2\sin^2\theta) \\
& -\frac{\cot\theta}{r^2}\left(g'' - \frac{2g}{r^2}\right) 2U\sin\theta\cos\theta \\
= \;& U\sin^2\theta\left(g''' - \left(\frac{v^2 2g' - 2g2r}{r^4}\right)\right)' \\
& -2\left(\frac{g''}{r^2} - \frac{2g}{r^4}\right) U\sin^2\theta = 0 \\
= \;& U\sin^2\theta\left(g'''' - \left(\frac{2g'}{r^2}\right)' + \left(\frac{4g}{r^3}\right)' - \frac{2g''}{r^2} + \frac{4g}{r^4}\right) = 0 \\
= \;& U\sin^2\theta\left(g'''' - \left(\frac{r^2 2g'' - 2r2g'}{r^4}\right)\right. \\
& \left. +4\left(\frac{r^3 g' - g3r^2}{r^6}\right) - \frac{2g''}{r^2} + \frac{4g}{r^4}\right) \\
= \;& 0 \\
\Rightarrow \quad & g'''' - \frac{4g''}{r^2} + \frac{8g'}{r^3} - \frac{8g}{r^4} = 0, \quad \text{put g} = \text{r}^\text{n}
\end{aligned}$$

$n(n-l)(n-2)(n-3) - 4n(n-1) + 8n - 8 = 0$
$(n-1)[n(n-2)(n-3) - 4n + 8] = 0$
$(n-1)(n-2)[n^2 - 3n - 4] = 0, \quad n = 1, 2, -1, 4$

$$\Rightarrow \psi = U \sin^2\theta \left[Ar^4 + Br^2 + Cr + \frac{D}{r}\right]$$

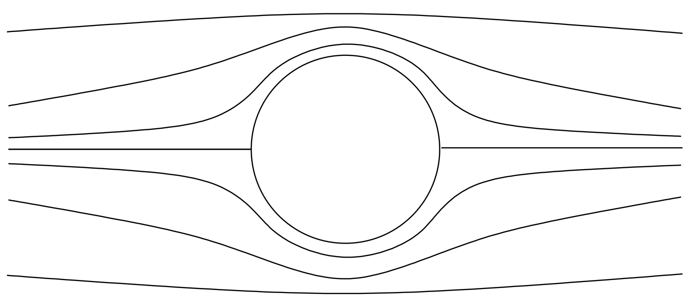
Conditions at ∞, $\Rightarrow A = 0, \quad B = \frac{1}{2}$

$$\text{Conditions at } r = a \Rightarrow \left.\begin{array}{r} \frac{a^2}{2} + Ca + \frac{D}{a} = 0 \\ a + c - \frac{D}{a^2} = 0 \end{array}\right\} \begin{array}{l} C = \frac{-3a}{4} \\ D = \frac{a^3}{4} \end{array}$$

$$\Rightarrow \psi = U \sin^2\theta \left[\frac{r^2}{2} - \frac{3ar}{4} + \frac{a^3}{4r}\right]$$

$$\begin{aligned} \Rightarrow u &= U\cos\theta \left[1 - \frac{3a}{2r} + \frac{a^3}{2r^3}\right] \\ v &= U\sin\theta \left[-1 + \frac{3a}{4r} + \frac{a^3}{4r^3}\right] \end{aligned}$$

(iii)



(Any reasonable symmetric effort will be accepted)

(iv) We have ignored$Re(\underline{q}.\nabla)\underline{q}$ in comparision to $\nabla^2\underline{q}$. Now for large r, $\underline{q} \sim 1$, $\nabla \sim \frac{1}{r}$ (non-dimensional).

$\Rightarrow Re\left(\frac{1}{r}\right) \ll \frac{1}{r^2}$ if slow flow is to hold.

$\Rightarrow Re \ll \frac{1}{r}$. But for fixed Re, (however small) we can always choose r large enough to violate this.

$\Rightarrow$ not valid for $r \geq O\left(\frac{1}{Re}\right)$