Question

A viscous incompressible fluid (having constant density ρ and constant kinematic viscosity ν) flows steadily in a pipe of constant cros-sectional are which has its generators parallel to the z-axis. There are no body forces. The cross-section of the pipe is denoted by D, so that the region occupied by the fluid is

$$\{(x, y, z) : (x, y) \in D, -\infty < z < \infty\}$$

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(a) By assuming a flow velocity $\underline{q}$ of the form $\underline{q} = (0, 0, w(x, y))$, show that w satisfies

$$\nabla^2 w = -\frac{P}{\mu}$$

where P is a constant that should be identified. Give the boundary consdtion that is satisfied by w on δD, the boundary of D.

(b) Verify that the function

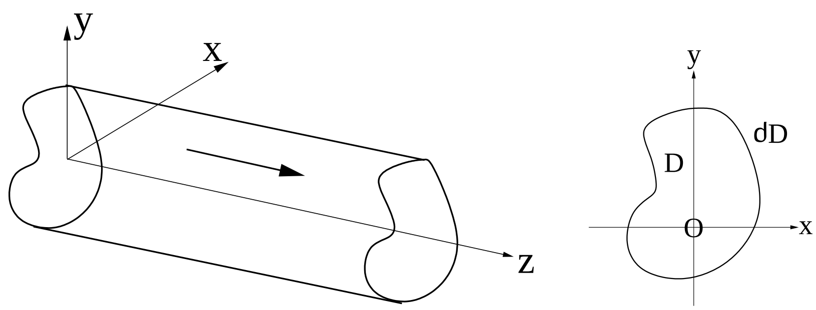
$$\phi(x, y) = K(x^2 - y^2/3)(y - a\sqrt{3})$$

satisfies $\nabla^2\phi = -P/\mu$ for a particular choice of K, which you should determine. By finding where $\phi = 0$ in the (x, y)-plane, identify the shape of the pipe through which the flow velocity is given by $\underline{q} = (0, 0, \phi(x, y))$.

(c) Determine the mass flow rate along the pipe.

Answer

(a)



Assume that $\underline{q} = (0, 0, w(x,y))$. Then

$$div(\underline{q}) = \frac{\partial}{\partial z}(w(x,y)) \equiv 0$$

Navier-Stokes give (note $\underline{q}_t = 0$ and $(\underline{q}.\nabla)\underline{q} = 0$)

$$\left.\begin{aligned} 0 &= \frac{-p_x}{\rho} + \nu(0) \\ 0 &= \frac{-p_y}{\rho} + \nu(0) \\ 0 &= \frac{-p_z}{\rho} + \nu(w_{xx} + w_{yy}) \end{aligned}\right\} \text{(no body forces)}$$

So $p_x = p_y = 0$ and $p = p(z)$ only. But from the 3rd equation, if $p = p(z)$ only and $w = w(x,y)$ only we must have $p_z = constant = -P$ say (thus identifying P).

$\Rightarrow$

$$0 = \frac{P}{\rho} + \frac{\mu}{\rho}(w_{xx} + w_{yy})$$

$$\Rightarrow \nabla^2 w = \frac{-P}{\mu} \quad ((x,y) \in D)$$

On ∂D we must have no-slip, so $w = 0$ on ∂D.

(b) Consider $\phi(x,y) = K(x^2 - \frac{y^2}{3})(y - a\sqrt{3})$

$$\begin{aligned}
\phi_x &= K2x(y - a\sqrt{3}) \\
\phi_y &= -(\frac{2Ky}{3})(y - a\sqrt{3}) + K(x^2 - \frac{y^2}{3}) \\
\Rightarrow \phi_y &= -Ky^2 + Kx^2 + 2Ka\sqrt{3}/3 \\
\phi_{xx} &= K2(y - a\sqrt{3})3 \\
\phi_{yy} &= -2Ky + 2Ka/\sqrt{3}
\end{aligned}$$

$\Rightarrow$

$$\begin{aligned}
\nabla^2\phi - \phi_{xx} + \phi yy &= 2Ky - 2Ka\sqrt{3} - 2Ky + 2Ka/\sqrt{3} \\
&= -4Ka/\sqrt{3}
\end{aligned}$$

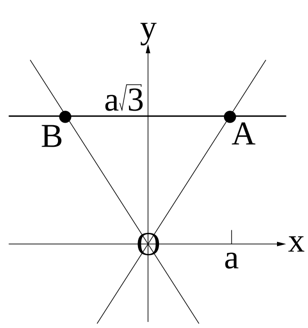
Thus $\nabla^2\phi = \dfrac{-P}{\mu}$ if $\dfrac{-P}{\mu} = \dfrac{-4Ka}{\sqrt{3}}$

$$\Rightarrow K = \frac{\sqrt{3}P}{4\mu a}$$

Now let us consider where $\phi = 0$ in the xy plane.

We have $\phi = 0$ when

$$\left.\begin{array}{ll}
(i) & y = a\sqrt{3} \\
(ii) & x = y/\sqrt{3} \\
(iii) & x = -y/\sqrt{3}
\end{array}\right\} \text{i.e. on 3 lines.}$$



The lines intersect at :
$O = (0, 0)$
$A = (a, \sqrt{3}a)$
$B = (-a, \sqrt{3}a)$

So the pipe shape is a triangle. Moreover,

$|AB| = 2a$, $|OA| = \sqrt{a^2 + (a\sqrt{3})^2} = 2A = |OB|$.

Thus the pipe is an EQUILATERAL TRIANGLE.

(c) To find the mass flow, need $\int_D \rho w \, dA$.

Since obviously symmetric about $x = 0$

$$
\begin{aligned}
M &= 2\rho \int_{y=0}^{\sqrt{3}a} \int_{x=0}^{y/\sqrt{3}} \frac{\sqrt{3}P}{4\mu a}(x^2 - y^2/3)(y - a\sqrt{3}) \, dx \, dy \\
&= \frac{2\rho\sqrt{3}P}{(3\sqrt{3})4\mu a} \int_0^a \sqrt{3}(\frac{y^3}{3} - y^3)(y - a\sqrt{3}) \, dy \\
&= \frac{2\rho P}{12\mu a} \int_0^{a\sqrt{3}} \frac{-2y^3}{3}(y - a\sqrt{3}) \, dy \\
&= \frac{-2P\rho}{18\mu a} \left[\frac{a^5(\sqrt{3})^5}{5} - \frac{\sqrt{3}a^5(\sqrt{3})^4}{4} \right] \\
&= \frac{Pa^4\sqrt{3}\rho}{20\mu}
\end{aligned}
$$