Question

A particle is projected along the positive y-axis so that initially, at time $t = 0$, its position is $y = 0$ and its velocity is $\frac{dy}{dx} = 2$. The equation of motion for the particle is described by

$$\frac{d^2y}{dx^2} = -8\cos^3(y)\sin(y).$$

(a) Calculate $\frac{df}{dy}$ where $f(y) = \cos^4(y)$

(b) Find expressions for the velocity of the particle as a function of y, and its position as a function of t.

(c) Sketch the graph of the position y as a function of time t for $t \geq 0$. Comment in the time taken for the particle to reach the point $y = \frac{\pi}{2}$

Answer

(a) Find $\frac{df}{dy}$ where $f(y) = \cos^4 y$.

Let $u = \cos y$ so that $\frac{du}{dy} = -\sin y$

Then $f(u) = u^4$ and $\frac{df}{du} = 4u^3$

$$\begin{aligned} \text{using the chain rule : } \frac{df}{dy} = \frac{df}{du} \times \frac{du}{dy} &= 4u^3(\sin y) \\ &= -\cos^3 y \sin y \end{aligned}$$

(b) Let $v = \frac{dy}{dt}$ so that

$$\frac{d^2y}{dt^2} = \frac{dV}{dt} = \frac{dV}{dy} \times \frac{dy}{dt} = \frac{dV}{dy}V$$

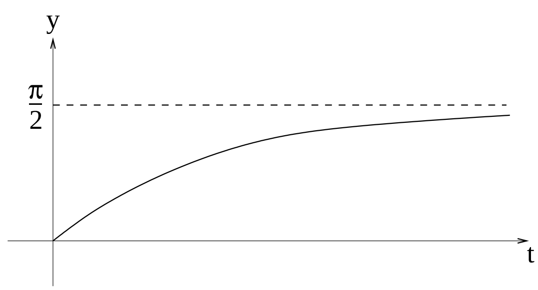
$$\begin{aligned} \text{Equation of motion becomes } V\frac{dV}{dy} &= -8\cos^3 y \sin y \\ \text{which is separable : } \int V dV &= 2\int -4\cos^3 y \sin y dy + C \\ \text{by part (a) } :\frac{1}{2}V^2 &= 2(\cos^4 y) + C \end{aligned}$$

$$\begin{aligned} \text{initial conditions : } V &= 2 \text{ and } y = 0 \text{when } t = 0 \\ \text{so } \frac{1}{2}(4) &= 2\cos^4(0) + C \Rightarrow c = 0 \\ \text{hence } V^2 &= 4\cos^4 y \\ \text{velocity as a function of y } v &= \sqrt{4\cos^4 y} = 2\cos^y \end{aligned}$$

(Take the positive square root to staify initial condition V = 2 when y = 0)

$$\begin{aligned}
\text{Now, } V = \frac{dy}{dt}\text{so } \frac{dy}{dt} &= 2\cos^2 y \\
\text{which is separable : } \int \frac{dy}{\cos^2 y} &= 2\int dt + K \\
\int \frac{dy}{\cos^y} &= \tan y \text{ standard integral} \\
\text{so } \tan y &= 2t + K \\
\text{using initial conditions } y = 0 \text{ when } t = 0 : \tan(0) &= 0 + K \\
\Rightarrow k &= 0 \\
\text{therefore } \tan y &= 2t \text{ and} \\
\text{position as a function of time : } y &= \tan^{-1}(2t)
\end{aligned}$$

(c) Plot position y against time t:



The graph is asymptotic to the line $y = \frac{\pi}{2}$.

Hence the particle **never** reaches $y = \frac{\pi}{2}$.