QUESTION

Use Laplace transforms to solve the partial differential equation

$$y_x + 2xy_t = 2x$$

given that $y(x,0) = y(0,t) = 1$.

ANSWER

$$y_x + 2xy_t = 2x \quad (1)$$
$$y(x,0) = 1 \quad (2)$$
$$y(0,t) = 1 \quad (3)$$

The Laplace transform of (1): $Y_x + 2x(Y - sy(x,0) = \frac{2x}{s}$

$Y_x + 2sxY = \frac{2x}{x} + 2x$

$\frac{d}{dx}\left(Ye^{sx^2}\right) = e^{sx^2}\left(\frac{2x}{s} + 2x\right)$

$Ye^{sx^2} = \int e^{sx^2}\left(\frac{2x}{s} + 2x\right)dx = e^{sx^2}\left(s^{-1} + s^{-2}\right) + C(s)$

$Y(x,s) = \frac{1}{s} + \frac{1}{s^2} + C(s)e^{-sx^2}$

To determine C(s), take the Laplace transform of (3)

$$\Rightarrow \quad Y(0,s) = \frac{1}{s}$$
$$\Rightarrow \quad C(s) = -\frac{1}{s^2}$$
$$\Rightarrow \quad Y(x,s) = \frac{1}{s} + \frac{1}{s^2}\left(1 - e^{-sx^2}\right)$$
$$\Rightarrow \quad y(x,t) = 1 + t - H(t - x^2)(t - x^2)$$
$$\Rightarrow \quad y(x,t) = \begin{cases} 1 + t \text{ for } t < x^2 \\ 1 + x^2 \text{ for } t > x^2 \end{cases}$$