QUESTION
For each of the following diophantine equations

(i) decide whether or not the solution exists,

(ii) if a solution exists, find the general solution,

(iii) Find the solution in which x takes the smallest possible positive integer value.

(a) $15x + 28y = 5$

(b) $18x + 27y = 7$

(c) $12x + 2y = 8$

ANSWER

(a) **(i)** gcd(15,28)=1, which divides 5, so solutions exist.

(ii) We seek a particular solution (x_0, y_0). If we can find it, then by th.1.11, the general solution is $(x_0 + 28t, y_0 - 15t)$ where $t \in Z$.

For a particular solution, we first use the Euclidean algorithm to solve $15x + 28y = 1$:

$$\begin{aligned} 28 &= 15.1 + 13 \\ 15 &= 13.1 + 2 \\ 13 &= 2.6 + 1 \\ 2 &= 2.1 + 0 \end{aligned}$$

Thus $\gcd(15, 28) = 1 = 13 - 2.6 = 13 - (15 - 13).6 = 13.7 - 15.6 = (28 - 15).7 - 15.6 = 28.7 - 15.13$. Multiplying by 5, $5 = 15 - 65 + 28.35$, so that $x_0 = -65$ and $y_0 = 35$ is a solution.

Hence $x = -65 + 28t,\ y = 35 - 15t$ $(t \in Z)$ is the general solution.

(iii) To make $x > 0$ we need $-65 + 28t > 0$, and the smallest $t \in Z$ for which this happens (and hance gives the smallest positive x) is 3. Thus the relevant solution is $x = 19,\ y = -10$.

(b) gcd(18,27)=9, which does not divide 7, so this equation has no solutions.

(c) **(i)** gcd(12,2)=2, which divides 8, so solutions exist.

(ii) If (x_0, y_0) is a particular solution, then by th.1.11, the general solution is $x = x_0 + \left(\frac{2}{2}\right)t,\ y = y_0 - \left(\frac{12}{2}\right)t$, i.e. $x = x_0 + t,\ y = y_0 - 6t$, where $t \in Z$.

To find a particular solution, we could follow the general policy, as we did in (a), but this time a short cut is available: if $12x+2y = 8$ then, on dividing by 2, $6x+y = 4$, and there is an obvious solution with $x = 0,\ y = 4$. The general solution is, therefore, $x = t,\ y = 4 - 6t$.

(iii) The solution where x takes the smallest positive value is given by $t = 1$. It is $x = 1,\ y = -2$.