QUESTION

Explain why, for any n, $n(n+1)$ is divisible by 2, and $n(n+1)(n+2)$ is divisible by 3.

Use these ideas to show that if n is an odd integer then $n^3 - n$ is divisible by 24.

ANSWER

Given $n \in Z$, either n is even or n is odd (i.e. $n = 2q$ or $n = 2q+1$). If n is odd, $n+1$ is even, so in all cases either n or $n+1$ is even, so $n(n+1)$ is even.

In a similar way, any $n \in Z$ can be written in one of the forms $3q$, $3q+1$, $3q+2$. If $n = 3q$, then $3|n$. If $n = 3q+1$, then $3|n+2$, and if $n = 3q+2$, then $3|n+1$. Thus in all cases, n divides one of n, $n+1$, $n+2$ and hence $3|n(n+1)(n+2)$, that is to say, 3 divides the product of any 3 consecutive integers.

Now $n^3 - n = n(n^2-1) = n(n-1)(n+1) = (n-1)n(n+1)$, so $n^3 - n$ is a product of 3 consecutive integers, and so, by the above, it is divisible by 3.

Now given n odd, we may write $n = 2q+1$, and then $(n-1)n(n+1) = 2q(2q+1)(2q+2) = 4(2q+1)q(q+1)$, and by the first part of the question $q(q+1)$ is even, so $n^3 - n$ is divisible by 8.

Thus $n^3 - n$ is divisible by 8 and 3, and hence by 24. (see cor.1.7).