QUESTION

What are the possible remainders when the fourth power of an integer is divided by 5?

ANSWER

Given integers m and n, we may divide n by m and get $n = qm + r$, with $0 \leq r < m$. If we raise this equation to the power k and use the binomial theorem we get

$$n^k = (qm+r)^k = (qm)^k + \left(\begin{array}{c} k \\ 1 \end{array}\right)(qm)^{k-1}r + \ldots + \left(\begin{array}{c} k \\ k-1 \end{array}\right)(qm)r^{k-1} + r^k.$$

and m divides every term on the right hand side except the last. Thus n^k and r^k have the same remainder om division by m.

Now for our particular case, $m = 5,\ k = 4$ and the possible remainders are 0,1,2,3,4. The remainders on dividing $0^4 = 0,\ 1^4 = 1,\ 2^4 - 16,\ 3^4 = 81$ and $4^4 = 256$ by 5 are 0,1,1,1,1 respectively, and so it follows that for any $n,\ n^4$ has remainder 0 or 1 on division by 5.

[Revision note: A much more rapid proof follows by using Fermat's Little Theorem-§4 of this course.]