QUESTION
Use the complex inversion formula to show that

$$\mathcal{L}^{-1}\frac{\sinh sx}{s^2\sinh sa} = \frac{xt}{a} + \frac{2a}{\pi^2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}\sin\frac{n\pi x}{a}\sin\frac{n\pi t}{a}.$$

ANSWER

$$f(s) = \frac{\sinh sx}{s^2\sinh sa}$$

This has simple poles at $s = \frac{n\pi i}{a}$ and a double pole at $s = 0$

$$\begin{aligned}
f(s)e^{st} &= \frac{\left(sx + \frac{(sx)^3}{3!} + \ldots\right)\left(1 + st + \frac{(st)^2}{2!} + \ldots\right)}{s^2\left(sa + \frac{(sa)^3}{s!}\right)} \\
&= \frac{x}{as^2}\frac{\left(1 + \frac{(3x)^2}{6} + \ldots\right)\left(1 + st + \frac{(st)^2}{2!} + \ldots\right)}{\left(1 + \frac{(sa)^2}{6}\right)} \\
&= \frac{x}{as^2}(1 + st + O(s^2)) \\
&\Rightarrow \ \mathrm{Res}(0) = \frac{xt}{a}
\end{aligned}$$

To find $\mathrm{Res}\left(\frac{n\pi i}{a}\right)$ use l'Hôpital's rule for simple poles

$$\begin{aligned}
f(s)e^{st} &= \lim_{s\to\frac{n\pi i}{a}}\frac{\sinh sxe^{st}}{s^2a\cosh sa + 2s\sinh a} \\
&= \frac{\sinh\frac{n\pi i}{a}xe^{\frac{n\pi i}{a}t}}{\left(\frac{n\pi i}{a}\right)^2 a\cosh n\pi i} \\
&= \frac{i\sin\frac{n\pi x}{a}e^{\frac{n\pi i}{a}t}}{-\left(\frac{n\pi}{a}\right)^2 a(-1)^n} \\
F(t) &= \frac{xt}{a} + \sum_{n=1}^{\infty}\mathrm{Res}\left(\frac{n\pi i}{a}\right) + \mathrm{Res}\left(-\frac{n\pi i}{n}\right) \\
&= \frac{xt}{a} + \sum_{n=1}^{\infty}\frac{2a}{n^2\pi^2}(-1)^n\sin\frac{n\pi x}{a}\sin\frac{n\pi t}{a}
\end{aligned}$$