QUESTION

$Y(x,t)$ satisfies the wave equation

$$Y_{tt} = c^2 Y_{xx},\ \ t > 0,\ \ 0 < x < a,$$

with

$$Y(x,0) = \frac{\beta x}{a},\quad Y_t(x,0) = 0,\quad Y(0,t) = 0,\quad Y_x(a,t) = 0.$$

(a) Show that

$$y(x,s) = \mathcal{L}Y(x,t) = \frac{\beta x}{a} - \frac{\beta c \sinh \frac{sx}{c}}{s^2 a \cosh \frac{sa}{c}}$$

(b) Use the complex inversion formula to find $Y(x,t)$.

ANSWER

(a) $s^2 Y - sY(x,0) - Y_t(x,0) = c^2 Y_{xx}$

$Y_{xx} - \frac{s^2}{c^2} Y = -\frac{s}{c^2}\frac{\beta x}{a}$

$Y = A(s) e^{\frac{s}{c}x} + B(s) e^{-\frac{s}{c}x} + \frac{\beta x}{as}$

$Y(0,t) = 0 \Rightarrow A(s) + B(s) = 0$

$Y_x(a,t) = 0 \Rightarrow A(s)\frac{s}{c}\left(e^{\frac{s}{c}a} + e^{-\frac{s}{c}a} + \frac{\beta}{as}\right) = 0$

$\Rightarrow Y(x,s) = \frac{\beta x}{as} - \frac{\beta c}{as^2}\frac{\sinh \frac{sx}{c}}{\cosh \frac{sa}{c}}$

(b) Simple pole at $s = 0$

$$\begin{aligned}
Ye^{st} &= \left(\frac{\beta x}{as} - \frac{\beta c}{as^2}\frac{\frac{sx}{c} + O(s^3)}{1 + O(s^2)}\right)(1 + st + O(st)) \\
&= \frac{\beta x}{as} - \frac{\beta c}{as^2}\frac{sx}{c} + O(1) \\
&\Rightarrow \ \mathrm{Res}(0) = 0
\end{aligned}$$

Simple poles at $s_n = (2n+1)\frac{c\pi i}{2a}$, use l'Hôpital's rule.

$$\begin{aligned}
\mathrm{Res}(Ye^{st}, s_n) &= \lim_{s \to s_n} \frac{\beta c}{a} \frac{e^{st}\sinh \frac{sx}{c}}{s^2 \frac{a}{c}\sinh\frac{a}{c}\sinh\frac{sa}{c} + 2s\cosh\frac{sa}{c}} \\
&= \frac{\beta c^2}{a^2}\frac{e^{s_n t}\sinh\frac{s_n x}{c}}{s_n^2(-1)^n} \\
Y(x,t) &= \sum_{n=1}^{\infty} \mathrm{Res}(s_n) + \mathrm{Res}(-s_n) \\
&= \sum_{n=1}^{\infty} \frac{8\beta}{\pi^2}\frac{(-1)^n}{(2n+1)^2}\sin\frac{(2n+1)\pi x}{2a}\cos\frac{(2n+1)\pi ct}{2a}
\end{aligned}$$