CONTINUED FRACTIONS
SYMMETRIC CONTINUED FRACTIONS

Let $[a_0, \ldots, a_n] = \frac{p_n}{q_n}$
Then

$$\begin{aligned}
p_n &= a_n p_{n-1} + p_{n-1}\ 0 < p_{n-2} < p_{n-1} \\
p_{n-1} &= a_{n-1} p_{n-2} + p_{n-3}\ 0 < p_{n-1} < p_{n-2} \\
&\vdots \\
p_2 &= a_2 p_1 + p_0 \\
p_1 &= a_1 p_0 + 1\ (p_0 = a_0) \\
p_0 &= a_0.1
\end{aligned}$$

This is the Euclidean algorithm for $(p_n, p_{n-1})$
So $\frac{p_n}{p_n-1} = [a_n, \ldots, a_o]$- the reverse of $\frac{p_n}{q_n}$
Also

$$\begin{aligned}
q_n &= a_n q_{n-1} + q_{n-2} \\
q_{n-1} &= a_{n-1} q_{n-2} + q_{n-3} \\
&\vdots \\
q_2 &= a_2 q_1 + 1\ q_0 = 1 \\
q_1 &= a_1.1
\end{aligned}$$

Again this is the Euclidean Algorithm so $\frac{q_n}{q_{n-1}} = [a_n, \ldots, a_1]$
Now suppose we have a symmetric continuous function $[a_0, a_1, a_2, \ldots, a_2, a_1, a_0]$ what can we say about the rational it gives rise to.
Theorem
A necessary and sufficient condition that an irreducible rational $\frac{P}{Q}$ $(P > Q > 1)$ should have a symmetric continued function with an even number of $a_i$'s is that $Q^2 + 1$ should be divisible by $P$.
A necessary and sufficient condition that an irreducible rational $\frac{P}{Q}$ $(P > Q > 1)$ should have a symmetric continued function with an odd number of $a_i$'s is that $Q^2 - 1$ should be divisible by $P$.
Proof
Necessity
Suppose $\frac{P}{Q} = [a_0, a_1, \ldots a_1, a_0] = \frac{p_n}{q_n}$ with $n+1$ entries. Since $(P, Q) = 1,\ P = p_n$ and $Q = q_n$. Because of symmetry

$$\frac{p_n}{q_n} = \frac{p_n}{p_{n-1}},$$

so $q_n = p_{n-1}$.
From the equation $p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$ we have

$$P q_{n-1} - (q_n)^2 = (-1)^{n-1}$$

$$P.q_{n-1} = Q^2 +)(-1)^{n-1}$$

so $Q^2 + (-1)^{n-1}$ is divisible by $P$.
Sufficiency
Suppose $Q^2 + \varepsilon = PQ'$ $\varepsilon = \pm 1$ $Q' \in N$
Expand $\frac{P}{Q}$ as a continued fraction

$$\frac{P}{Q} = [a_0, \ldots, a_n] = \frac{p_n}{q_n}$$

where $n$ is chosen so that $(-1)^{n-1} = \varepsilon$. This is possible because of the ambiguity at the end of a finite continued fraction. Now $(P, Q) = 1$ so $P = p_n$ $Q = q_n$ and so $q_n^2 + \varepsilon = p_n Q'$ also $q_n p_{n-1} + (-1)^{n-1} = p_n q_{n-1}$
Subtracting gives $q_n(q_n - p_{n-1}) = p_n(Q' - q_{n-1})$
Hence $q_n - p_{n-1}$ is divisible by $p_n$ since $(p_n, q_n) = 1$.
But $p_n > q_n > 0$ and $p_n > p_{n-1} > 0$
so $p_n > |q_n - p_{n-1}|$
So, since $p_n | q_n - p_{n-1}$, $q_n - p_{n-1} = 0$
so $\frac{p_n}{q_n} = [a_0, \ldots a_n] = \frac{p_n}{p_{n-1}}$
but $\frac{p_n}{p_{n-1}} = [a_n, \ldots, a_0]$. So the continued fraction is symmetric.