CONTINUED FRACTIONS
RATIONAL NUMBERS

We now insist that $a_n \in N$ for all n. Every finite continued fraction represents a rational number.
Furthermore every rational number can be represented as a finite continued fraction, from the Euclidean algorithm.
Now suppose that

$$x = a_0 + \frac{1}{a_1+} \dots \frac{1}{a_N} \qquad x \in Q$$

If we insist that the last a_N is > 1 then this is unique. Otherwise the only other representation is

$$x = a_0 + \frac{1}{a_1+} + \dots \frac{1}{a_{N-1}+} \frac{1}{(a_N - 1)+} \frac{1}{1}$$

Proof
Let $x \in Q$ and suppose $x = [a_0; a_i \dots a_N] = [b_0; b_1 \dots b_M]$
Let $a'_n = [a_n; a_{n+1} \dots a_N]$ $b'_n[n_n; n_{n+1} \dots b_M]$
So

$$\begin{aligned} x &= [a_0; a_1 \dots a_{n-1}, a'_n] \; 1 \leq n \leq N \\ &= [b_0; b_1 \dots b_{n-1}, b'_n] \; 1 \leq n \leq M \end{aligned}$$

For $n < N$

$$a'_n = a_n + \frac{1}{a'_{n+1}}, \; (a_{n+1}, > 1)$$

so that we must have $a_n = [a'_n]$
Similarly for $n < M$ $b_n = [b'_n]$
Now we must have $a_0 = b_0 = [x]$, so suppose $a_1 = b_1, \dots a_{n-1} = b_{n-1}$
so $\frac{p_{n-1}}{q_{n-1}} = [a_0; \dots, a_{n-1}] = [b_0; \dots b_{n-1}]$
Thus

$$x = \frac{a'_n p_{n-1} + p_{n-2}}{a'_n q_{n-1} + q_{n-2}} = \frac{b'_n p_{n-1} + p_{n-2}}{b'_n q_{n-1} + q_{n-2}}$$

Cross multiplying gives

$$b'_n(p_{n-2}q_{n-1} - p_{n-1}q_{n-2}) = a'_n(p_{n-2}q_{n-1} - p_{n-1}q_{n-2})$$

so $a'_n = b'_n$ since the term in the bracket $=(-1)^n \neq 0$.

So, since $a_n = [a'_n]$ and $b_n = [b'_n]$ $a_n = b_n$.
So $a_n = b_n$ for all n, and when one expansion terminates so does the other.
Note that $a_n = [a'_n]$ requires $a_n + 1 > 1$. If $a'_{n+1} = 1$ then $a_n = [a'_n] - 1$ and $a_{n+1} = 1$, and the expansion terminates. All previous a'_i are > 1, otherwise the expansion would have terminated earlier.
Linear Diophantine Equations.
Because of the ambiguity at the end of the expansion of a rational number, we can always an expansion $[a_o; \ldots a_N]$ with N odd.
Let $x = \frac{a}{b}$. So $\frac{a}{b} = \frac{p_n}{q_n}$

$$p_N q_{N-1} - q_N p_{N-1} = (-1)^{N-1} = 1$$

i.e.

$$a q_{N-1} - b p_{N-1} = 1$$

So $x = p_{N-1}$ $y = q_{N-1}$ is a positive integer solution of the equation $ax - by = 1$
Notice that this shows $(P_{N-1}, q_{N-1}) = 1$
In fact since $p_n q_{n-1} - q_n p_{n-1} = (-1)^{n-1}, (p_n, q_n) = 1$ for any n
So when we "add up" a continued fraction we always obtain the rational answer in it's lowest terms.
The following result concerns approximation.
Let α be a real algebraic number of degree n. So

$$f(\alpha) = a_n \alpha_n + a_{n-1} \alpha^{n-1} + \ldots + a_0 = 0, \ a_n \neq 0, \ a_i \in Z$$

Now $\exists M, \forall x \in (\alpha - 1, \alpha + 1) |f'(x)| < M$ Suppose $\frac{p}{q} \neq alpha$ is an approximation to α, close enough to be in $(\alpha - 1, \alpha + 1)$, and nearer to α than any other root of $f(x) = 0$, so $f\left(\frac{p}{q}\right) \neq 0$.

$$\left| f\left(\frac{p}{q}\right) \right| = \frac{|a_0 p^n + a_1 p^{n-1} q + \ldots|}{q^n} \geq \frac{1}{q^n}$$

since the numerator is a positive integer.

$$F\left(\frac{p}{q}\right) = f\left(\frac{p}{q}\right) - f(\alpha) = \left(\frac{p}{q} - \alpha\right) f'(\xi) \ MUT$$

with ξ between $\frac{p}{q}$ and α so

$$\left| \frac{p}{q} - \alpha \right| = \left| \frac{f\left(\frac{p}{q}\right)}{f'(\xi)} \right| > \frac{1}{M q^n} = \frac{K}{q^n}$$

We have already shown that if α is irrational then there are infinitely many solutions of

$$\left|\frac{p}{q} - \alpha\right| < \frac{1}{q^2}$$

α is said to be approximable to order 2.
Now let $\alpha = 0.110001000\ldots = \frac{1}{10} + \frac{1}{10^{2!}} + \frac{1}{10^{3!}} + \ldots$
Let $n_0 \in N$ and let $n > n_0$. Let

$$\alpha_n = \frac{p}{10^{n!}} = \frac{p}{q}$$

be the sum of the first n terms of the series

$$0 < \alpha - \frac{p}{q} = \frac{1}{10^{(n+1)!}} + \frac{1}{10^{(n+2)!}} + \ldots$$

$$< \frac{2}{10^{(n+1)!}} < \frac{2}{q^{n+1}} < \frac{2}{q^{n_0}}$$

so α is approximable to order n_0, for any n_0.
Hence α is transcendental.
The theorem above shows that an algebraic number of degree n is not approximable to order $\nu = n$

A Thue	(1901)	improved this to $\nu = \frac{n}{2} + 1$
C L Siegel	(1921)	improved this to $\nu > min_{1 \le s \le n-1, s \in N} \left(\frac{n}{s+1} + s\right)$
F J Dyson	(1947)	improved this to $\nu > 2\sqrt{n}$
K F Roth	(1955)	improved this to $\nu = 2 + \varepsilon$ for all $\varepsilon > 0$

This result is best possible.