CONTINUED FRACTIONS
PERIODIC CONTINUED FRACTIONS

Consider first a purely periodic continued fraction

$$\begin{aligned} \alpha &= [\overline{a_0; a_1, a_2 \ldots, a_n}] \\ &= [a_0; a_1, \ldots, a_n, \alpha] \end{aligned}$$

So $\alpha = \frac{\alpha p_n + p_{n-1}}{\alpha q_n + q_{n-1}}$
so $\alpha^2 q_n + \alpha(q_{n-1} - p_n) - p_{n-1} = 0$
This has two roots

$$\begin{aligned} \alpha &= \frac{(p_n - q_{n-1}) + \sqrt{(p_n - q_{n-1})^2 + 4q_n p_{n-1}}}{2q_n} > 0 \\ \overline{\alpha} &= \frac{(p_n - q_{n-1}) - \sqrt{(p_n - q_{n-1})^2 + 4q_n p_{n-1}}}{2q_n} < 0 \end{aligned}$$

Furthermore the LHS of the quadratic equation is $-p_{n-1} < 0$ for $\alpha = 0$ and $(q_n - q_{n-1}) + (p_n - p_{n-1} > 0$ for $\alpha = -1$. Thus $\alpha > 0$ and $-1 < \overline{\alpha} < 0$.
Quadratic irrationals with this property are termed reduced.
This is related to reduced quadratic forms as Gauss defined them.
Now if we have a periodic continued function where the period starts at stage $k + 1$

$$\begin{aligned} \beta &= [b_0; b_1, \ldots b_k, \overline{a_0, a_1, \ldots a_n}] \\ \beta &= \frac{\alpha p_k + p_{k-1}}{\alpha q_k + q_{k-1}} \end{aligned}$$

Since α is a quadratic irrational, so is β.
Thus any periodic continued fraction represents a quadratic irrational.
Example
$\beta = [2, 3, \overline{10, 1, 1, 1}]$ $\alpha = [\overline{10, 1, 1,}]$
$2 + \frac{1}{3} = \frac{7}{3}$ so $\beta = \frac{7\alpha+2}{3\alpha+1}$
To evaluate α

n	-1	0	1	2	3
a_n		10	1	1	1
p_n	1	10	11	21	32
q_n	0	1	1	2	3

So

$$\begin{aligned}\alpha &= \frac{32\alpha+21}{3\alpha+2}\\ 3\alpha^2+(2-32)\alpha-21 &= 0\\ \alpha^2-10\alpha-7 &= 0\end{aligned}$$

The positive root is $\alpha = 5+\sqrt{32} = 5+4\sqrt{2}$.
So $\beta = \frac{35+28\sqrt{2}+2}{15+12\sqrt{2}+1} = \frac{20-\sqrt{2}}{8}$
We now prove the convers.
Theorem
A continued fraction which represents a quadratic irrational is periodic (Lagrange)
Proof
Let $\alpha = \frac{P+\sqrt{D}}{Q} > 0$ be a positive quadratic irrational.
Now $\alpha = \frac{Pm+\sqrt{Dm^2}}{Qm} = \frac{P'+\sqrt{D'}}{Q'}$
$\frac{D'-P'^2}{Q'} = m.\frac{D-P^2}{Q} \in Z$ for suitable m.
So suppose w.l.o.g. $\alpha_0 = \frac{P_0+\sqrt{D}}{Q_0}$ and $Q_0|D-P_0^2$ $D_0, q_0 \in Z$
$\alpha_0 = a_0 + \frac{1}{\alpha_1}$ so $\frac{1}{\alpha_1} = \frac{\sqrt{D}+P_0-a_0Q_0}{Q_0}$ and

$$\alpha_1 = \frac{Q_0}{\sqrt{D}+P_0-a_0Q_0} = \frac{\sqrt{D}-a_0Q_0-P_0}{\frac{1}{Q_0}[D-(a_0Q_0-P_0)^2]} = \frac{\sqrt{D}+P_1}{Q_1}$$

(note that $Q_0|[]$) where $P_1 = a_0Q_0 - P_0$

$$Q_1 = \frac{D-(a_0Q_0-P_0)^2}{Q_0} = \frac{D-P_1^2}{Q_0}$$

So $Q_0 = \frac{D-P_1^2}{Q_1}$ i.e. $Q_1|D-P_1^2$
so this divisibility property is preserved through the continued fraction algorithm.
At the kth stage we therefore have $\alpha_k = \frac{\sqrt{D}+P_k}{Q_k}$ where

$$\begin{aligned}P_k &= a_{k-1}Q_{k-1} - P_{k-1}\\ Q_k &= \frac{(D-P_k^2)}{Q_{k-1}}\\ a_k &= [\alpha_k]\end{aligned}$$

We now show that the process eventually produces reduced quadratic irrationals.

$$\alpha_0 = [a_0, a_1, \dots a_n \alpha_{n+1}] = \frac{\alpha_{n+1} p_m + p_{n-1}}{\alpha_{n+1} q_n + q_{n-1}}$$

If we denote $\overline{\alpha}$ the quadratic conjugate of α then we have

$$\begin{aligned} \overline{\alpha_0} &= \frac{\overline{\alpha_{n+1}} p_n + p_{n-1}}{\overline{\alpha_{n+1}} q_n + q_{n-1}} \\ \overline{\alpha_{n+1}} &= \frac{-\overline{\alpha_0} q_{n-1} + p_{n-1}}{\overline{\alpha_0} q_n - p_n} = -\frac{q_{n-1}}{q_n} \left(\frac{\overline{\alpha_0} - \frac{p_{n-1}}{q_{n-1}}}{\overline{\alpha_0} - \frac{p_n}{q_n}} \right) \end{aligned}$$

Now as $n \to \infty$

$$\frac{\overline{\alpha_0} - \frac{p_{n-1}}{q_{n-1}}}{\overline{\alpha_0} - \frac{p_n}{q_n}} \to \frac{\overline{\alpha_0} - \alpha_0}{\overline{\alpha_0} - \alpha_0} = 1$$

so for all n sufficiently large

$$\overline{\alpha_{n+1}} = -\frac{q_{n-1}}{q_n}(1 + \varepsilon_n) \ |\varepsilon| \text{ small}$$

so $\overline{\alpha_{n+1}} < 0$ for all n sufficiently large.
If $\overline{\alpha_k} < 0$ then from $\alpha_k = a_k + \frac{1}{\alpha_{k+1}}$ we conclude

$$\frac{1}{\overline{\alpha_{k+1}}} = \overline{\alpha_k} - a_k < -a_k \leq -1$$

So we conclude that for all n sufficiently large $-1 < \overline{\alpha_n} < 0$, and since $\alpha_n > 1$, α_n is a reduced quadratic irrational.
Now if $\alpha = \frac{P+\sqrt{D}}{Q}$ is reduced then $\alpha - \overline{\alpha} > 0$ and $\alpha + \overline{\alpha} > 0$, so $\frac{2\sqrt{D}}{Q} > 0$ and $\frac{2\sqrt{P}}{Q} > 0$ so $Q > 0$ and so $P > 0$
Also $\overline{\alpha} < 0$ so $P - \sqrt{D} < 0$ i.e. $P < \sqrt{d}$.
Also $\alpha > 1$ so $Q < P + \sqrt{D}$
If $N = [\sqrt{D}]$ we have $0 < P \leq N$ and $0 < Q \leq 2N$
so there is only a finite set of reduced quadratic irrationals associated with a given $\sqrt{D}$. So at some stage in the process we eventually have

$$\alpha_h = \alpha_k$$

i.e.

$$[a_h; a_{h+1}, a_{h+2}, \dots] = [a_k; a_{k+1}, a_{k+1}, \dots]$$

and by uniqueness the a's are equal, so we have periodicity, where the shortest period is $\leq |k - h|$
Note. If $N = [\sqrt{D}]$ there are at most $2N^2$ RQI so the period is $\leq 2N^2 \sim 2D$

(i)

$$\alpha = \frac{\sqrt{2}-20}{-8} = [2, 3, \overline{10, 1, 1, 1}]$$

$N = 1$ but the period is $> 2N$?

This is because α is not in a form which satisfies the divisibility condition. We need to write

$$\alpha = \frac{\sqrt{2m^2}-20m}{-8m} \ (m > 0)$$

and we need $8m|2m^2 - (20m)^2$ i.e. $8|2m - 400m$

$m = 4$ is the first solution. So we must write $\alpha = \frac{\sqrt{32}-80}{-32}$

we then use the recursive scheme for the p's and Q's to obtain the complete quotients.

k	0	1	2	3	4	5	6
P_k	-80	16	5	5	2	2	5
Q_k	-32	7	1	7	4	7	1
a_k	2	3	10	1	1	1	

$$P_k = a_{k-1}Q_{k-1} - P_{k-1}$$

$$Q_k = \frac{D - P_k^2}{Q_{k-1}}$$

$$a_k = \left[\frac{P_k + \sqrt{D}}{Q_k}\right] = \left[\frac{P_k + [\sqrt{D}]}{Q_k}\right]$$

(ii) The number of RQI is about $\leq$ about $2D$ astablished above. This can be improved. In 1971 KEH showed that this number is $O(\sqrt{D}\log D)$

We now want to see where the period starts. We have seen that a purely periodic continued fraction represents a RQI.

Theorem

A RQI has a purely periodic continued fraction (Galois)

Proof

Let α_0 be reduced, so $\alpha_0 > 1$ and $-1 < \overline{\alpha_0} < 0$

Now $\alpha_0 = a_o + \frac{1}{\alpha_1}$ and $\alpha_1 > 1$

Also $\overline{\alpha_0} = a_0 + \frac{1}{\overline{\alpha_1}}$ and $\overline{\alpha_0} < 0$ so $\frac{1}{\overline{\alpha_1}} = -a_0 + \overline{\alpha_0} < -a_0 \leq -1$

so $-1 < \overline{\alpha_1} < 0$ and so α_1 is reduced. Continuing gives

$\alpha_n = a_n + \frac{1}{\alpha_{n+1}}$ and $-\frac{1}{\overline{\alpha_{n+1}}} = a_n - \overline{\alpha_n}$ and $0 < -\overline{\alpha_n} < 1$

Thus

$$a_n = [\alpha_n] = \left[-\frac{1}{\overline{\alpha_{n+1}}} \right]$$

Suppose that the period for the continued fraction begins with a_m $m \geq 1$. Then

$$\alpha_0 = [a_0, a_1 \dots a_{m-1} \overline{a_m a_{m+1} \dots a_{m+k-1}}]$$

period k and $a_{m-1} \neq a_{m+k-1}$
Because of periodicity however, $|alpha_m = \alpha_{m+k}$ and so $-\frac{1}{\overline{\alpha_m}} = -\frac{1}{\overline{\alpha_{m+k}}}$
Taking the integer part gives $a_{m-1} = a_{m+k-1}$
So the period must start at a_0.
Given $\alpha_0 > 1$, if $\overline{\alpha_0} < -1$, α_0 is not reduced and so its contu=inued fraction is not purely periodic. But $\overline{\alpha_0} < 0 \Rightarrow \alpha_1$ reduced, so if $\overline{\alpha_0} < -1$ then the continued fraction has just one term before the period begins.
Suppose that α_0 has just one term before the period begins

$$\alpha_0 = [a_0; \overline{a_1, \dots, a_k}]$$

α_0 is not reduced. Now $\alpha_1 = \alpha_{k+1}$ and $a_0 = a_0 + \frac{1}{\alpha_1}$ $\alpha_k = a_k + \frac{1}{\alpha_{k+1}} = a_k + \frac{1}{\alpha_1}$ so

$$\begin{aligned} \overline{\alpha_0} &= a_0 + \frac{1}{\overline{\alpha_1}} \\ \overline{\alpha_k} &= a_k + \frac{1}{\overline{\alpha_1}} \end{aligned}$$

So, since $-1 < \overline{a_k} < 0$, if $\overline{\alpha_0} < -1$, $a_0 < a_k$ and if $\overline{\alpha_0} > 1$ $a_0 > a_k$.
Naturally if $\overline{\alpha_0} > 1$ the continued fraction for α_0 may have more then one entry in its a cyclic part.