CONTINUED FRACTIONS
IRRATIONAL NUMBERS

Irrational numbers
Terminology:
If $\alpha = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\dots$ then the a_i are called partial quotients.

$$a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\dots\frac{1}{a_n} = \frac{p_n}{q_n}$$

is called the n-th convergent
If we write

$$\alpha = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\dots\frac{1}{a_{n-1}+}\frac{1}{\alpha_n}$$

α_n is called a complete quotient.
To develop a number $\alpha \in R-Q$ as a continued fraction, we use the following recursive scheme (we take $\alpha > 0$)
$\alpha = a_0 + \frac{1}{\alpha_1}$ where $a_0 = [\alpha]$ and so $\alpha_1 > 1$
$\alpha + n = a_n + \frac{1}{\alpha_{n+1}}$ where $a_n = [\alpha_n]$ and so $\alpha_n > 1$.
We need to give a meaning to the infinite expression

$$[a_0; a_1, a_2 \dots]$$

and to associate it with α.
Suppose we take this development of an irrational and truncate it so

$$\frac{p_n}{q_n} = [a_0; a_1, \dots, a_n]$$

and

$$\alpha = [a_0; a_1 \dots a_n \alpha_{n+1}]$$

$$\begin{aligned} \alpha - \frac{p_n}{q_n} &= \frac{\alpha_{n+1}p_n + p_{n-1}}{\alpha_{n+1}q_n + q_{n-1}} - \frac{p_n}{q_n} \\ &= \frac{(-1)^n}{q_n(\alpha_{n+1}q_n + q_{n-1})} \to 0 \text{ as } n \to \infty \end{aligned}$$

So every number has a continued fraction expression - finite if rational and infinite if irrational. It is unique apart from the choice at the end of a finite continued fraction.

Further more every continued fraction does converge. We have seen that the even convergents form a sequence bounded above, and the odd convergents form a sequence bounded below.
Also $\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_n q_{n-1}} \to 0$ as $n \to \infty$ as $q_n \to \infty$ as $n \to \infty$
Hence there is a 1-1 correspondence between sequences of natural numbers and irrational numbers. The Cantor diagonal argument is even easier in this case than with decimals, to prove that the irrationals are uncountable.
List irrationals $\alpha_1, \dots \alpha_n \dots$
Let α be the irrational whose continued fraction has its n-th partial quotient 1 more than that of α_n, for each n.
We can say a bit more about convergence to α, as follows:
Again writing $\alpha = [a_0; a_1 \dots a_n \alpha_{n+1}]$

$$|q_n \alpha - p_m| = \frac{1}{\alpha_{n+1} q_n + q_{n-1}}$$

Now $\alpha_{n+1} q_n + q_{n-1} > a_{n+1} q_n + q_{n-1} = q_{n+1}$
However $\alpha_{n+1} q_n + q_{n-1} < (a_{n+1} + 1) q_n + q_{n-1} = q_{n+1} + q_n < q_{n+2}$ so

$$\frac{1}{q_{n+2}} < |q_n \alpha - p_n| < \frac{1}{q_{n+1}}$$

$$\frac{1}{q_{n+3}} < |q_{n+1} \alpha - p_{n+1}| < \frac{1}{q_{n+2}}$$

so $|q_{n+1}\alpha - p_{n+1}| < |q_n \alpha - p_n|$
Divide LHS by q_{n+1} and RHS by q_n ($q_{n+1} > q_n$) to give

$$\left|\alpha - \frac{p_{n+1}}{q_{n+1}}\right| < \left|\alpha - \frac{p_n}{q_n}\right|$$

So the convergents get successively nearer to α, alternating either side of α.
An aside:
Consider now the set of all real numbers whose first $(n+1)$ partial quotients are fixed.

$$\alpha = [a_0; \dots a_n, \alpha_{n+1}] \text{ so } 1 \le \alpha_{n+1} < \infty,\ 0 < \frac{1}{\alpha_{n+1}} \le 1$$

Consider $\alpha_x = [a_o; \dots a_n, x]$ $\alpha_y = [a_0, \dots a_n, y]$

$$\begin{aligned} \alpha_x - \alpha_y &= \frac{x p_n + p_{n-1}}{x q_n + q_{n-1}} - \frac{y p_n + p_{n-1}}{y q_n + q_{n-1}} \\ &= \frac{(x p_n + + p_{n-1})(y q_n + q_{n-1}) - (y p_n + p_{n-1})(x q_n + q_{n-1})}{(x q_n + q_{n-1})(y q_n + q_{n-1})} \end{aligned}$$

$$
\begin{aligned}
&= \frac{(x-y)(p_n q_{n-1} - p_{n-1} q_n)}{(xq_n + q_{n-1})(yq_n + q_{n-1})} \\
&= \frac{(x-y)(-1)^{n-1}}{(xq_n + q_{n-1})(yq_n + q_{n-1})}
\end{aligned}
$$

so if n is even α_x is a decreasing function of x and if n is odd α_x is an increasing function of x.
So with $0 < x \leq 1$, α_x occupies an interval, with one end point (included)

$$[a_0; \ldots, a_n, 1] = \frac{p_n + q_{n-1}}{q_n + q_{n-1}}$$

and the other end point (excluded)

$$[a_0; \ldots a_n] = \frac{p_n}{q_n}$$

whose length is

$$\left| \frac{p_n}{q_n} - \frac{p_n + p_{n-1}}{q_n + q_{n-1}} \right| = \frac{1}{q_n(q_n + q_{n-1})}$$