CONTINUED FRACTIONS

Continued functions are related to the Euclidean algorithm. For example $\frac{10}{7}$

$$\begin{aligned} 10 &= 1.7+3 \\ 7 &= 2.3+1 \\ 3 &= 3.1 \end{aligned}$$

can be written as

$$\frac{10}{7} = 1 + \frac{3}{7} \qquad \frac{7}{3} = 2 + \frac{1}{3}$$

so

$$\frac{10}{7} = 1 + \frac{1}{2+\frac{1}{3}}$$

We can generalise steps like this, so that instead of taking a rational number like $\frac{10}{7}$ we could take an irrational number α, and write

$$\alpha = a_0 + \frac{1}{\alpha_1}, \qquad a_0 = [\alpha],\ \alpha_1 > 1$$

$$\alpha_1 = a_1 + \frac{1}{\alpha_2}$$

etc.

For example

$$\begin{aligned} \sqrt{6} &= 2 + \left(\sqrt{6}-2\right) \\ \frac{1}{\sqrt{6}-2} = \frac{\sqrt{6}+2}{2} &= 2 + \frac{\sqrt{6}-2}{2} \\ \frac{2}{\sqrt{6}-2} = 2.\frac{\sqrt{6}+2}{2} = \sqrt{6}+2 &= 4 + \left(\sqrt{6}-2\right) \end{aligned}$$

the process then repeats.

So

$$\sqrt{6} = 2 + \frac{1}{2+\frac{1}{4+\frac{1}{2+\frac{1}{4+\frac{1}{\cdot}}}}}$$

Periodic from the first step onwards.

We clearly nee a better notation. Several have been in use over the years.

$$\begin{aligned}
\sqrt{6} &= 2.\&\frac{1}{2.}\&\frac{1}{4.}\&\frac{1}{2.}\&\frac{1}{4.}\&\ldots \\
&= 2+\frac{1}{2+}\frac{1}{4+}\frac{1}{2+}\cdots \\
&= [2;2,4,2,4\ldots] \\
&= [2;\overline{2,4}]
\end{aligned}$$

To work with continued functions I need to establish some fundamental formulae.
Consider the continued function

$$a_0+\frac{1}{a_1+}\frac{1}{a_2+}\cdots$$

where the a_i for the present could be thought of as variables (real, complex...). If we consider the finite fraction

$$a_0+\frac{1}{a_1+}\frac{1}{a_2}\cdots\frac{1}{a_n}$$

this will be a rational function in the variables, which we shall write as $\frac{p_n}{q_n}$.
Evaluating the first few values
$a_0 = \frac{p_0}{q_0}$ so $p_0 = a_0,\ q_0 = 1$
$a_0 + \frac{1}{a_1} = \frac{a_0a_1+1}{a_1}$ so $p_1 = a_0a_1 + 1$ $q_1 = a_1$

$$a_0+\frac{1}{a_1+}\frac{1}{a_2}=\frac{a_0a_1a_2+a_0+a_2}{a_1a_2+1}=\frac{p_2}{q_2}$$

$$\begin{aligned}
a_0+\frac{1}{a_1+}\frac{1}{a_2+}\frac{1}{a_3} &= \; = \frac{a_0a_1a_2a_3+a_0a_3+a_2a_3+a_0a_1+1}{a_1a_2a_3+a_3+a_1} \\
&= \frac{a_3(a_0a_1a_2+a_0+a_2)+a_0a_1+1}{a_3(a_1a_2+1)+a_1}=\frac{p_3}{q_3}
\end{aligned}$$

so $p_3 = a_3p_2 + p_1$ $q_3 = a_3q_2 + q_1$
This pattern generalises, and we have

$$\begin{aligned}
p_{n+1} &= a_{n+1}p_n + p_{n-1} \\
q_{n+1} &= a_{n+1}q_n + q_{n-1}
\end{aligned}$$

Proof By induction

$$\frac{p_n}{q_n} = a_0 + \frac{1}{a_1+}\frac{1}{a_2+}\dots\frac{1}{a_n} = \frac{a_n p_n - 1 + p_{n-2}}{a_n q_{n-1} + q_{n-2}}$$

Now if we replace a_n by $a_n + \frac{1}{a_{n+1}}$ we obtain $\frac{p_{n+1}}{q_{n+1}}$
so

$$\begin{aligned}
\frac{p_{n+1}}{q_{n+1}} &= \frac{\left(a_n + \frac{1}{a_{n+1}}\right) p_{n-1} + p_{n-2}}{\left(a_n + \frac{1}{a_{n+1}}\right) q_{n-1} + q_{n-2}} \\
&= \frac{a_n p_{n-1} + p_{n-2} + \frac{p_{n-1}}{a_{n+1}}}{a_n q_{n-1} + q_{n-1} + \frac{q_{n-1}}{a_{n+1}}} \\
&= \frac{a_{n+1} p_n + p_{n-1}}{a_{n+1} q_n + q_{n-1}}
\end{aligned}$$

The formulae we developed initially gives
$n = 2$ $\quad p_2 = a_2 p_1 + p_0,\ q_2 = a_2 q_1 + q_0$
$n = 1$ would read $p_1 = a_1 p_0 + p_{-1},\ q_1 = a_1 q_0 + q_{-1}$
Now this requires

$$\begin{aligned}
a_0 a_1 + 1 &= a_1 a_0 + p_{-1},\ p_{-1} = 1 \\
a_1 &= a_1.1 + q_{-1},\ q_{-1} = 0
\end{aligned}$$

So if we conventionally set $p_{-1} = 1$ $q_{-1} = 0$ then these formulae are fine for $n = 1, 2, \dots$.
If the a_i are positive integers we can use these recurrence relations to work out $\frac{p_n}{q_n}$ successively, without having to "add up from the back end" each time.
eg. $3 + \frac{1}{7+}\frac{1}{15+}\frac{1}{1+}\frac{1}{292+}\frac{1}{1+}\dots$

n	-1	0	1	2	3	4	5
a_n		3	7	15	1	292	1
p_n	1	3	22	333	355	103933	104288
q_n	0	1	7	105	113	33102	33215

Now as a decimal $\pi = 3.141592653897932\dots$

$\frac{p_n}{q_n}$	$n = 1$	3.
	$n = 2$	3.142857...
	$n = 3$	3.14150943...
	$n = 4$	3.14159292...
	$n = 5$	3.14159265301...

Since the a_i are positive integers, we have

$$p_n \geq p_{n-1} + p_{n-2},\ q_n \geq q_{n-1} + q_{n-1}$$

with equality if and only if $a_n = 1$
The minimum possible values for q_n and p_n therefore occur if all the a_n's are 1.
In that case
$p_{-1} = 1,\ p_0 = 1$ and $p_n = p_{n-1} + p_{n-1}$
$q_0 = 1$ and $q_1 = 1$ and $q_n = q_{n-1} + q_{n-2}$
So q_n is the n th term in the Fibonacci sequence and p_n is the $(n+1)$th term in the Fibbonacci sequence.
i.e. for $1 + \frac{1}{1+}\frac{1}{1+}\frac{1}{1+}\ldots$ we have

n	-1	0	1	2	3	4	5
a_n		1	1	1	1	1	1
p_n	1	1	2	3	5	8	13
q_n	0	1	1	2	3	5	8

$\frac{p_n}{q_n} \longrightarrow$ golden ratio.
If $x = 1 + \frac{1}{1+}\frac{1}{1+}\ldots$ then (without worrying about convergence) $x = 1 + \frac{1}{x}$
So $x^2 - x - 1 = 0,\ x = \frac{1 \pm \sqrt{5}}{2}$ and $x > 0$ so $x = \frac{1+\sqrt{5}}{2}$.
More identities

1.
$$p_n q_{n-1} - p_{n-1} q_n = (-1)^{n-1}$$

2.
$$p_n q_{n-1} - p_{n-2} q_n = (-1)^n a_n$$

1. Again by induction

$$\begin{aligned} & p_{n+1} q_n - p_n q_{n+1} \\ = & (a_n p_n + p_{n-1}) q_n - p_n (a_n q_n + q_{n-1}) \\ = & -(p_n q_{n-1} - p_{n-1} q_n) \end{aligned}$$

$n = 1 : p_1 q_0 - p_0 q_1 = (a_0 a_1 + 1).1 - a_0.a_1 = 1$

2.

$$\begin{aligned} & p_n q_{n-2} - p_{n-2} q_n \\ = & (a_n p_{n-1} + p_{n-2}) q_{n-2} - p_{n-2} (a_n q_{n-1} + q_{n-2}) \\ = & a_n (p_{n-1} q_{n-2} - q_{n-1} p_{n-2}) = a_n (-1)^n \end{aligned}$$

by 1.

These formula can also be written in the form

1. $$\frac{p_n}{q_n} - \frac{p_{n-1}}{q_{n-1}} = \frac{(-1)^{n-1}}{q_n q_{n-1}}$$

2. $$\frac{p_n}{q_n} - \frac{p_{n-2}}{q_{n-2}} = \frac{(-1)^n a_n}{q_n q_{n-2}}$$

Now suppose the a_n's are positive real numbers. For n even, since the q's are all positive

$$\frac{p_{n-2}}{q_{n-2}} < \frac{p_n}{q_n} < \frac{p_{n-1}}{q_{n-1}}$$

For n odd

$$\frac{p_{n-1}}{q_{n-1}} < \frac{p_n}{q_n} < \frac{p_{n-2}}{q_{n-2}}$$

So this gives

$$\frac{p_0}{q_0} < \frac{p_2}{q_2} < \frac{p_4}{q_4} < \frac{p_6}{q_6} < \ldots < \frac{p_5}{q_5} < \frac{p_3}{q_3} < \frac{p_1}{q_1}$$