Question

a) Prove that, for $|x| < 1$

$$\int_0^\infty \frac{\sin t}{e^t - x} dt = \sum_{n=1}^\infty \frac{x^{n-1}}{n^2+1}.$$

b) The function $J_m(x)$ is defined by

$$J_m(x) = \sum_{n=0}^\infty \frac{(-1)^n}{n!(n+m)!} \left(\frac{x}{2}\right)^{m+2n}.$$

Prove that for all real values of a,

$$2\int_0^\infty J_m(2ax) x^{m+1} e^{-x^2} dx = a^m e^{-a^2}.$$

[In both cases each step in the calculations should be carefully explained, and a clear statement given of any theorems applied, together with a proof that any conditions for the application of theorems are satisfied.]

Answer

In both examples we use the following theorem, which is a consequence of Lebesgue's Dominated Convergence Theorem.

Theorem A Suppose that $\{f_n\}$ is a sequence of integrable functions. Then $\int \sum f_n = \sum \int f_n$ provided either $\sum \int |f_n|$ or $\int \sum |f_n|$ is finite. (The two conditions are in fact equivalent.)

a) $$\int_0^\infty \frac{\sin t}{e^t - x} dt = \int_0^\infty \frac{\sin t}{e^t(1 - \frac{x}{e^t})} dt$$

Now for $t \geq 0$ $\left|\frac{x}{e^t}\right| \leq |x| < 1$ and so

for all $t \geq 0$ $\dfrac{1}{1 - \frac{x}{e^t}} = 1 + \dfrac{x}{e^t} + \dfrac{x^2}{e^{2t}} + \dfrac{x^3}{e^{3t}} + \cdots$

Thus for $|x| < 1$

$$\int_0^\infty \frac{\sin t}{e^t - x} dt = \int_0^\infty \sum_{n=1}^\infty x^{n-1} \frac{\sin t}{e^{nt}} dt \qquad (1)$$

Now $\displaystyle\sum_{n=1}^\infty \int_0^\infty \left| x^{n-1} \frac{\sin t}{e^{nt}} \right| dt$

$$\leq \sum_{n=1}^\infty \int_0^\infty |x|^{n-1} \frac{1}{e^{nt}} dt = \sum_{n=1}^\infty \frac{|x|^{n-1}}{n} < \infty \quad \text{for } |x| < 1.$$

We can thus apply theorem A to interchange the order of integration and summation in equation (1). We then have

$$\int_0^\infty \frac{\sin t}{e^t - x} dt = \sum_{n=1}^\infty x^{n-1} \int_0^\infty \frac{\sin t}{e^{nt}} dt = \sum_{n=1}^\infty \frac{x^{n-1}}{n^2 + 1}$$

(The integral $\displaystyle\int_0^\infty \frac{\sin t}{e^t - x} dt$ is easily evaluated as $\frac{1}{n^2+1}$)

b) $J_m(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} \left(\frac{x}{2}\right)^{m+2n}$

$$J_m(2ax) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} a^{m+2n} x^{m+2n}$$

$$2\int_0^{\infty} J_m(2ax) x^{m+1} e^{-x^2} dx$$

$$= 2\int_0^{\infty} \sum_{n=0}^{\infty} \frac{(-1)^n}{n!(n+m)!} a^m.a^{2n}.x^{m+2n}.x^{m+1}.e^{-x^2} dx$$

$$= a^m \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{n!} \int_0^{\infty} \frac{2x^{2(m+n)+1}}{(n+m)!} e^{-x^2} dx$$

(*)

$$= a^m \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{n!} \frac{1}{(n+m)!} \int_0^{\infty} t^{m+n} e^{-t} dt$$

$$= a^m \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{n!} \frac{1}{(n+m)!} \Gamma(m+n+1)$$

$$= a^m \sum_{n=0}^{\infty} \frac{(-1)^n a^{2n}}{n!} \qquad [\Gamma(m+n+1) = (m+n)!]$$

$$= a^m e^{-a^2}$$

The inversion of order of summation and integration at (*) is justified by the steps after (*) which (with the factor $(-1)^n$ removed, and a replaced by $|a|$) show that $\sum \int |f_n| < \infty$, thus enabling theorem A to be applied.