Question

Discuss Riemann and Lebsgue integrability on (0,1) of the functions f and g defined below. In each case calculate the integrals, where they exist.

(a) $f(x) = 1$ if x is irrational,

$f(x) = \sin \dfrac{1}{x}$ if x is rational.

(b) $g(x) = x$ if x is irrational,

$g(x) = \dfrac{b+1}{c}$ if x is rational, and $x = \dfrac{b}{c}$, where b and c have no common factors.

(In each case you should prove any assertions you make concerning continuity of function. Conditions for integrability should be stated but not proved.)

Answer

(a)

$$\begin{array}{lll} f(x) = 1 & x \text{ irrational} & 0 < x < 1 \\ f(x) = \sin \frac{1}{x} & x \text{ irrational} & 0 < x < 1 \end{array}$$

f is continuous if $x = \dfrac{1}{(2n+\frac{1}{2})\pi}$ $n \epsilon \mathbf{N}$ and discontinuous elsewhere (proof is standard analysis.)

Thus the set of discontinities of f in (0,1) has measure 1 and so f is not R - integrable. However $f = 1$ a.e. in (0,1) and s f is Lebesgue integrable and

$$L \int_0^1 f = 1$$

(b) g is continuous at all irrational values (proof is standard analysis)

So the set of discontinities has measure zero and so g is R - integrable. Thus g is L-integrable. $g(x) = x$ a.e. and so

$$R \int_0^1 g = L \int_0^1 g = L \int_0^1 x = \frac{1}{2}$$