Question

Let f be a positive bounded measurable function with domain $[0,1]$. Using the definition of the Lebesgue integral of f over $[0,1]$, in terms of sequences of simple functions, prove that

$$\int_0^1 f = m_2(E),$$

where m_2 denotes Lebesgue measure in the plane, and E is the region under the graph of f, i.e.

$$E = \{(x,y)|0 \leq x \leq 1; 0 \leq y \leq f(x)\}$$

Answer

In defining $\int_0^1 f$ we prove that a positive measurable function f can be expressed as a limit of a monotonic increasing sequence of simple functions. A simple function is one taking a finite number of values on measurable subsets of $[0,1]$. If $\{Ei\}_{i=1}^n$ is a measurable partition of $[0,1]$ then a simple function is one of the form

$g(x) = \sum_{i=1}^n c_i X_{Ei}(x),$ and we define

$\int_0^1 g = \sum_{i=1}^n c_i m(Ei)$

We then prove that if $\{f_n\}$ and $\{g_n\}$ are two increasing sequences of functions converging to f then,

$\lim_{n\to\infty} \int_0^1 f_n = \lim_{n\to\infty} \int_0^1 g_n$

We thus define $\int_0^1 f = \lim_{n\to\infty} \int_0^1 f_n$ for any sequence $\{f_n\}$ of simple functions converging to f.

We now use this to prove the stated result.

Let $Q_{p,s} = \{x\epsilon[0,1)|\frac{p-1}{2^s} \leq f(x) < \frac{p}{2^s}\} \quad p = 1, 2, \cdots, 2^{2s}$

Since f is bounded, if s is large enough then $\{Q_{p,s}\}_{p=1}^{2^{2s}}$ forms a partition of $[0,1)$.

We consider a sequence of simple functions defined for such s.

$f_s(x) = \frac{p-1}{2^s}, \quad \text{for } x\epsilon Q_{p,s} \quad p = 1, 2, \cdots 2^{2s}$

Clearly f_s is a simple function. It can be shown that $f_s \to f$.

Let $S_{p,s} = \{(x,y) | x \epsilon Q_{p,s} \text{ and } 0 \leq y \leq \frac{p}{2^s}\}$
Then $E \subseteq \bigcup_p S_{p,s}$ for all s. Thus $E \subseteq \bigcap_s \bigcup_p S_{p,s}$
Now let $(x,y) \epsilon \bigcap_s \bigcup_p S_{p,s}$ then
for all s, there exists p, such that $x \epsilon Q_{p,s}$ and $0 \leq y \leq \frac{p}{2^s}$
for all s, there exists p, such that $\frac{p-1}{2^s} \leq f(x) < \frac{p}{2^s}$ and $0 \leq y \leq \frac{p}{2^s}$
for all s, $0 \leq y \leq f(x) + \frac{1}{2^s}$ i.e. $y \leq f(x)$
Thus $\bigcap_s \bigcup_p S_{p,s} = E$
Now let $T_{p,s} = \{(x,y) | x \epsilon Q_{p,s} \text{ and } 0 \leq y \leq \frac{p-1}{2^s}\}$
Then $m_2 \left(\bigcup_p T_{p,s} \right) = \sum_p \frac{p-1}{2^s} m(Q_{p,s}) = \int_0^1 f_s$
$\bigcup_p S_{p,s} - \bigcup_p T_{p,s} = \bigcup_p \{(x,y) | x \epsilon Q_{p,s} \text{ and } \frac{p-1}{2^s} < y \leq \frac{p}{2^s}\}$
Thus $m_2 \left(\bigcup_p S_{p,s} - \bigcup_p T_{p,s} \right) = \sum_p \frac{m(Q_{p,s})}{2^s} = \frac{1}{2^s}$
Thus $m_2 \left(\bigcup_p S_{p,s} \right) = m_2 \left(\bigcup_p T_{p,s} \right) + \frac{1}{2^s}$
$E = \bigcap_s \bigcup_p S_{p,s}$ and $\cup_p S_{p,s}$ has finite measure as f is bounded, also $\{\bigcup_p S_{p,s}\}_s$ is a decreasing sequence of sets.
Thus $m_2(E) = \lim_s m_2(\bigcup S_{p,s})$
$= \lim_s \left[m_2 \left(\bigcup_p T_{p,s} \right) + \frac{1}{2^s} \right] = \lim_s m_2 \left(\bigcup_p T_{p,s} \right)$
$= \lim_s \int_0^1 f_s = \int_0^1 f$
Hence the result.