Question

a) Let T be the set of all real numbers in $(0,1)$ whose decimal expansion does not contain the digit 7. Prove that T has Lebesgue measure zero.

b) Let S be a measurable subset of $[0,1]$ with the property that if $x \epsilon S$, and if the decimal expansion of y differs from that of x in only a finite number of places then $y \epsilon S$.

Show that if a and b are real numbers in $[0,1]$ with terminating decimal expansions, and $a < b$, then

$$m(S \cap [a,b]) = m(S) \cdot (b-a).$$

Deduce that this result is true for all intervals $[a,b] \subseteq [0,1]$.

Hence or otherwise prove that S has measure 0 or 1.

Answer

a) Let T be the set of all real numbers in $(0,1)$ whose decimal expansion does not contain the digit 7.

Then $T \subseteq \displaystyle\bigcup_{\substack{r_i = 0 \\ r_i \neq 7 \\ i = 1, \cdots n}}^{10} \left[\sum_{i=1}^{n-1} \frac{r_i}{10^i} + \frac{r_n}{10^n}, \ \sum_{i=1}^{n-1} \frac{r_i}{10^i} + \frac{r_n + 1}{10^n} \right]$

Thus $m(T) \leq \displaystyle\sum_{\substack{r_i = 0 \\ r_i \neq 7 \\ i = 1, \cdots n}}^{10} m \left[\sum_{i=1}^{n-1} \frac{r_i}{10^i} + \frac{r_n}{10^n}, \ \sum_{i=1}^{n-1} \frac{r_i}{10^i} + \frac{r_n + 1}{10^n} \right]$

$$= \sum_{\substack{r_i = 0 \\ r_i \neq 7 \\ i = 1, \cdots n}}^{10} \frac{1}{10^n} = \frac{9^n}{10^n} < \epsilon \text{ if } n \text{ is large enough.}$$

Therefore $m(T) = 0$.

b) Suppose $a = \dfrac{m_1}{10^{n_1}}, \quad b = \dfrac{m_2}{10^{n_2}}$

Let $n = \max(n_1, n_2)$ then we can write $a = \dfrac{l_1}{10^n}, \quad b = \dfrac{l_2}{10^n}$

$$[a, b] = \bigcup_{i=l_1}^{l_2 - 1} \left[\frac{i}{10^n}, \ \frac{i+1}{10^n}\right]$$

[These abutting intervals have just one point in common with their neighbours, there are only a finite number of them, and so additivity still holds.]

Now let $x \epsilon S \cap \left[\frac{i}{10^n}, \ \frac{i+1}{10^n}\right]$ and consider $S \cap \left[\frac{j}{10^n}, \ \frac{j+1}{10^n}\right]$

The number $y = x + \frac{j-i}{10^n}$ will belong to $\left[\frac{j}{10^n}, \ \frac{j+1}{10^n}\right]$ and will differ at most the first n decimal places from the expansion of x.

Thus we shall have $y \epsilon S \cap \left[\frac{j}{10^n}, \ \frac{j+1}{10^n}\right]$.

The reverse process can also be applied, and so this proves that
$S \cap \left[\frac{i}{10^n}, \ \frac{i+1}{10^n}\right]$ and $S \cap \left[\frac{j}{10^n}, \ \frac{j+1}{10^n}\right]$
are simply translates of one another.

Thus $m(S) = 10^n m \left(S \cap \left[\frac{i}{10^n}, \ \frac{i+1}{10^n}\right]\right)$ for all i,

and so $m(S \cap [a, b[) = (l_2 - l_1) m \left(S \cap \left[\dfrac{i}{10^n}, \ \dfrac{i+1}{10^n}\right]\right)$

$= \frac{l_2 - l_1}{10^n} m(S) = (b - a) m(S)$

Now if a and b are any real numbers ($a < b$) we can express a as a decreasing sequence of terminating decimals $\{a_n\}$ and b as an increasing sequence of terminating decimals $\{b_n\}$ with $b_n > a_n$.

Then $(a, b) = \bigcup_{n=1}^{\infty} [a_n, b_n]$

and so $S \cap (a, b) = \bigcup_{n=1}^{\infty} S \cap [a_n, b_n]$.

Thus $m(S \cap [a, b]) = m(S \cap (a, b)) = m \left(\bigcup_{n=1}^{\infty} S \cap [a_n, b_n] \right)$

$= \lim_{n \to \infty} m(S \cap [a_n, b_n]) = \lim(b_n - a_n)m(S) = (b - a)m(S)$

Now let I_n be any system of intervals satisfying $\bigcup I_n \supseteq S$.

Then $m(S) = m(S \cap \bigcup I_n) = m(\bigcup S \cap I_n)$

$\leq \sum m(S \cap I_n) = m(S) \sum |I_n|$

i.e. $m(S) \leq m(S) \sum |I_n|$

so either $m(S) = 0$ or $\sum |I_n| \geq 1$

i.e. $m(S) \geq 1$ but $S \subseteq [0, 1]$,

so either $m(S) = 0$ or $m(S) = 1$