Question

Let E be a bounded measurable subset of the plane. Let $H(t)$ be the half-plane defined by

$$H(t) = \{(x, y)|x \leq t\}.$$

The function $f(t)$ is defined for all real t by

$$f(t) = m(E \cap H(t)),$$

where m denotes Lebesgue measure in the plane.

Prove that f is a continuous function, and deduce that there is a line in the plane parallel to the y-axis which bisects E in the sense that a subset of E of measure $\frac{1}{2}m(E)$ lies on each side of the line.

Give an example of a set E for which the set $f^{-1}(\frac{1}{2}m(E))$ contains more than one point.

Show that for all sets E, either $f^{-1}(\frac{1}{2}m(E))$ is a single point or it is a closed interval.

Answer

E is bounded and so E is a subset of some square S with sides of length l parallel to the co-ordinate axes.
If $t_1 < t_2$ then $E \cap H(t_1) \subseteq E \cap H(t_2)$ and so $f(t_1) \leq f(t_2)$
Then $f(t_2) - f(t_1) = m(E \cap H(t_2)) - m(E \cap H(t_1))$
$= m[(E \cap H(t_2)) - (E \cap H(t_1))] = m[E \cap (H(t_2) - H(t_1)]$
$\leq m(S \cap (H(t_2) - H(t_1))) = l(t_2 - t_1)$.
Thus f is continuous everywhere (in fact f is uniformly continuous).

Since $E \subseteq S$, if t is large and negative $f(t) = 0$ and if t is large and positive $f(t) = m(E)$. Thus, by the intermediate value theorem for continuous functions there is a number t_0 such that

$$f(t_0) = \frac{1}{2}m(E)$$

so to the left of the line $x = t_0$ lies a portion of E of measure $\frac{1}{2}m(E)$ (since the measure of $E \cap l$ is zero for any line l) and also to the right of this line lies a portion of E of measure $\frac{1}{2}m(E)$.

Example. If E_1 is the unit disc centre -2 and E_2 is the unit disc centre $+2$, and $E = E_1 \cup E_2$ then $f(-1) = f(+1) = \frac{1}{2}m(E)$.

Now suppose $f^{-1}(\frac{1}{2}m(E))$ is not a single point.
Let $a = \inf\{x | x \epsilon f^{-1}(\frac{1}{2}m(E))\}$ $b = \sup\{x | x \epsilon f^{-1}(\frac{1}{2}m(E))\}$, then $a < b$. We can find a sequence $x_n \to a$ such that $x_n \epsilon f^{-1}(\frac{1}{2}m(E))$ i.e. $f(x_n) = \frac{1}{2}m(E)$. Since f is continuous $f(a) = \lim f(x_n) = \frac{1}{2}m(E)$ i.e. $a \epsilon f^{-1}(\frac{1}{2}m(E))$. Similarly $b \epsilon f^{-1}(\frac{1}{2}m(E))$. Since f is an increasing function.

$f(x) = \frac{1}{2}$ for all x satisfying $a \leq x \leq b$ and so $f^{-1}(\frac{1}{2}) = [a, b]]$.