Question

Prove that if $J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$

then $\int_0^{\infty} e^{-ax} J_0(x) dx = \frac{1}{\sqrt{1+a^2}}$

Answer

$J_0(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n}$

$\int_0^{\infty} e^{-ax} J_0(x) = \int_0^{\infty} e^{-ax} \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \left(\frac{x}{2}\right)^{2n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \frac{1}{4^n} \int_0^{\infty} e^{-ax} x^{2n} dx$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \frac{1}{4^n} \frac{1}{a^{2n+1}} \int_0^{\infty} e^{-y} x^{2n} dy$

$= \sum_{n=0}^{\infty} \frac{(-1)^n}{(n!)^2} \frac{1}{4^n} \frac{1}{a^{2n}} \Gamma(2n+1)$

$= \frac{1}{a} \sum_{n=0}^{\infty} \frac{(-1)^n (2n)!}{(n!)^2} \frac{1}{4^n} \frac{1}{a^{2n}} = \frac{1}{\sqrt{1+a^2}}$