Question

Prove that $\int_0^\infty \frac{x^{a-1}}{e^x - 1} dx = \Gamma(a)\zeta(a)$

Answer

$$\int_0^\infty \frac{x^{a-1}}{e^x - 1} = \int_0^\infty \frac{x^{a-1}e^{-x}}{1 - e^{-x}} dx = \int_0^\infty x^{a-1}e^{-x} \sum_{n=0}^{\infty} e^{-nx} dx$$
$$= \sum_{n=0}^{\infty} \int_0^\infty x^{a-1}e^{-(n+1)x} dx = \sum_{n=0}^{\infty} \frac{1}{(n+1)^a} \int_0^\infty t^{a-1}e^{-t} dt = \Gamma(a)\zeta(a)$$