Question

Prove that $\int_0^1 \frac{x^p}{1-x} \log\left(\frac{1}{x}\right) dx = \sum_{n=0}^{\infty} \frac{1}{(p+n+1)^2}$

Answer

$$\int_0^1 \frac{x^p}{1-x} \log\left(\frac{1}{x}\right) = \int_0^1 x^p \log\frac{1}{x} \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \int_0^1 x^{p+n} \log\frac{1}{x}$$

$$= \sum_{n=0}^{\infty} \frac{1}{(p+n+1)^2}, \quad \text{since}$$

$$\int_0^1 x^\alpha \log\frac{1}{x} = \frac{1}{\alpha+1}\left[x^{\alpha+1}\log\frac{1}{x}\right]_0^1 + \frac{1}{\alpha+1}\int_0^1 x^{\alpha+1}\frac{x}{x^2} = \frac{1}{(\alpha+1)^2}$$