Question

(a) Show that all the roots of the equation

$$(1+x)^{2n+1} = (1-x)^{2n+1}$$

are given by

$$\pm i \tan\left(\frac{k\pi}{2n+1}\right) \qquad k = 0, 1, 2, \cdots, n$$

By putting $n = 2$ show that

$$\tan^2\left(\frac{\pi}{5}\right)\tan^2\left(\frac{2\pi}{5}\right) = 5.$$

(b) If $w = 2z + z^2$ show that the circle $|z| = 1$ corresponds to a cardioid in the w-plane.

Answer

(a)

$$\begin{aligned}
(1+x)^{2n+1} &= (1-x)^{2n+1} \\
\text{So } \frac{1+x}{1-x} &= e^{\frac{2\pi i}{2n+1}k} \\
x &= \frac{e^{\frac{2\pi i}{2n+1}k} - 1}{e^{\frac{2\pi i}{2n+1}k} + 1} \\
&= \frac{e^{\frac{\pi i k}{2n+1}} - e^{-\frac{\pi i k}{2n+1}}}{e^{\frac{\pi i k}{2n+1}} + e^{-\frac{\pi i k}{2n+1}}} \\
&= i \tan\frac{\pi k}{2n+1} \qquad k = -n, \cdots, n \\
&= \pm i \tan\frac{\pi k}{2n+1} \qquad k = 0, \cdots, n
\end{aligned}$$

Putting $n = 2$. The equation reduces to $x(x^4 + 10x^2 + 5) = 0$.

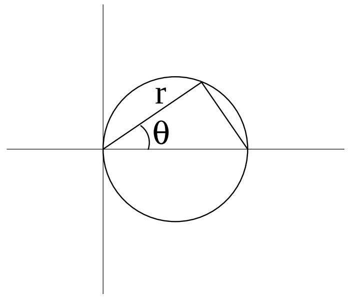
So the product of the non-zero roots is 5.

i.e. $\tan^2\left(\frac{\pi}{5}\right)\tan^2\left(\frac{2\pi}{5}\right) = 5$.

(b) $w = 2z + z^2$

$w + 1 = (z+1)^2$

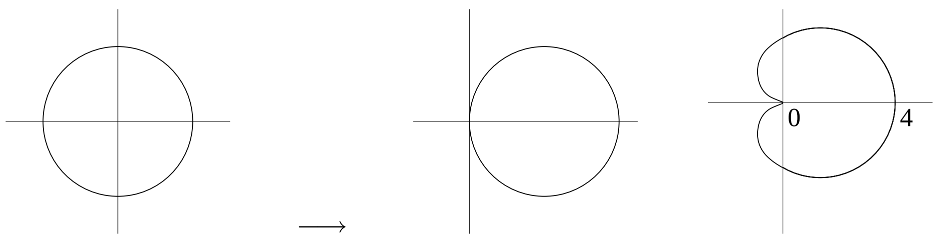
If z lies on the unit circle $z + 1$ lies on the circle centre 1 radius 1



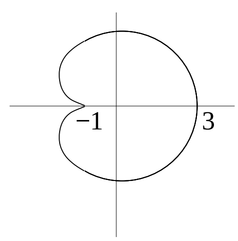
$r = 2\cos\theta$

So $r^2 = 4\cos^2\theta$

Let $w + 1 = \rho e^{i\phi}$ $z + 1 = re^{i\theta}$ the $\rho = r^2$ and $\phi = 2\theta$
So $\rho^2 = 16\cos^2\frac{\phi}{2} = 8(1 + \cos\phi)$ which is a cardioid.



z $\qquad \zeta = z + 1 \qquad \omega = w + 1$



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