Question

This question concerns the frequently-traded derivative product known as a "Perpetual American Put".

(a) Define the terms PERPETUAL, AMERICAN and PUT.

(b) Explain why, for a Perpetual American Put, the Black-Scholes equation reduces to

$$\frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0$$

where σ denotes volatility, r denotes interest rate, S denotes the asset value and V denotes the value of the option.

(c) Given a simple arbitrage argument to show that the option value can never be less than the early-exercise price and thus

$$V \geq \max(E - S, 0).$$

Explain why V must also satisfy $V \to 0$ as $S \to \infty$.

(d) Denote the optimal exercise price (i.e. the price which should automatically trigger exercise) by S^*. Explain why

(i) $V(S^*) = E - S^*$

(ii) $\partial V / \partial S^* = 0$

Hence or otherwise, show that the value of the option is given by

$$V(S) = \left(\frac{E}{1 + \frac{\sigma^2}{2r}} \right)^{1+2r/\sigma^2} \frac{\sigma^2}{2r} S^{-2r/\sigma^2}.$$

Answer

(a)

PERPETUAL:- The option has no expiry and 'goes on forever'.

AMERICAN:- The option may be exercised at any time of the holder's choosing.

PUT:- The payoff at exercise is determined by $\max(E - S, 0)$

(b) The standard Black-Scholes equation is

$$V_t + \frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0,$$

but if the option is perpetual then by definition it cannot have a value that depends on time. Thus $V_t = 0$ and Black-Scholes becomes

$$\frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0$$

(c) Suppose it were that case that $V < \max(\mathrm{E} - \mathrm{S}, 0)$.

Then buy the option (cost V) and exercise straight away (receiving $E - S$). Then the payoff will be

$$-V + (E - S) = (E - S) - V > 0$$

which is a risk free profit.

Thus $V \geq \max(\mathrm{E} - \mathrm{S}, 0)$.

Also, the option is a put, and so profits if the share price reduces. If $S \to \infty$ therefore the option value must become zero.

(d) If the situation $S = S^*$ ensures that we exercise the option, then when this happens the value of the option must be the payoff.

Thus $V(S^*) = E^*_S$.

Also, The OPTIMAL EXERCISE PRICE must, by definition, be optimal and thus maximize the option value. So regarded as a function of S and S^*, at optimal conditions

$$\frac{\partial V}{\partial S^*}(S, S^*) = 0$$

Now we must solve

$$\frac{1}{2}\sigma^2 S^2 V_{SS} + rSV_S - rV = 0$$

subject to

$$\begin{aligned} V \to 0 \quad as & \quad S \to \infty \\ V(S^*) &= E - S^* \\ \frac{\partial V}{\partial S^*} &= 0 \end{aligned}$$

To solve the ODE (Euler's equation), try

$$V \propto S^n$$

(n to be determined)

$$\begin{aligned} \frac{1}{2}\sigma^2 n(n-1) + rn - r &= 0 \\ \Rightarrow n &= 1, \quad n = -2r/\sigma^2 \end{aligned}$$

Thus $V = AS + BS^{-2r/\sigma^2}$, (A,B arbitrary constants).

Now we need $V \to 0$ as $S \to \infty \Rightarrow A = 0$.

Also, since $V(S^*) = E_S^*$

$$\begin{aligned} B(S^*)^{-2r/\sigma^2} &= E - S^* \\ \Rightarrow B &= (S^*)^{2r/\sigma^2}(E - S^*) \end{aligned}$$

Thus

$$V = (E - S^*)(S^*)^{2r/\sigma^2} S^{-2r/\sigma^2}$$

Now

$$\frac{\partial V}{\partial S^*} = -\left(\frac{S^*}{S}\right)^{\frac{2r}{\sigma^2}} + (E - S^*)\frac{2r}{\sigma^2}\left(\frac{S^*}{S}\right)^{\frac{2r}{\sigma^2}} \frac{1}{S^*}$$

$$\begin{aligned} \Rightarrow S^* &= (E - S^*_{\frac{2r}{\sigma^2}} \\ &= \frac{2Er}{\sigma^2\left(1 + \frac{2r}{\sigma^2}\right)} \\ &= \frac{2Er}{\sigma^2 + 2r} \end{aligned}$$

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\begin{aligned}
\Rightarrow V &= \left(\frac{2Er}{\sigma^2+2r}\right)^{\frac{2r}{\sigma^2}} S^{-\frac{2r}{\sigma^2}} \left(E - \frac{2Er}{\sigma^2+2r}\right) \\
&= \left(\frac{E}{1+\frac{\sigma^2}{2r}}\right)^{\frac{2r}{\sigma^2}+1} S^{-\frac{2r}{\sigma^2}} E \left(\frac{\sigma^2}{\sigma^2+2r}\right) \\
&= \left(\frac{E}{1+\frac{\sigma^2}{2r}}\right)^{\frac{2r}{\sigma^2}} S^{-\frac{2r}{\sigma^2}} \frac{E\sigma^2}{2r} \left(\frac{2r}{\sigma^2+2r}\right) \\
&= \left(\frac{E}{1+\frac{\sigma^2}{2r}}\right)^{\frac{2r}{\sigma^2}+1} \frac{\sigma^2}{2r} S^{-\frac{2r}{\sigma^2}}
\end{aligned}
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