Question

In this question YOU MAY ASSUME that the value V of a European Vanilla Call is given by

$$V - SN(d_1) - Ee^{-r(T-t)}N(d_2)$$

where

$$\begin{aligned} d_1 &= \frac{\log(S/E) + \left(r + \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\ d_2 &= \frac{\log(S/E) + \left(r - \frac{1}{2}\sigma^2\right)(T-t)}{\sigma\sqrt{T-t}}, \\ N(p) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{p} e^{-q^2/2}\, dq \end{aligned}$$

and S, E, t, T, σ and r denote respectively the asset price, the strike price, the time, the expiry, the volatility and the interest rate.

(a) Show that

$$\frac{\partial}{\partial S}(N(d_1)) = \frac{1}{\sqrt{2\pi(T-t)}} \frac{e^{-d_1^2/2}}{\sigma S}$$

and calculate

$$\frac{\partial}{\partial S}(N(d_2)).$$

Hence show that the DELTA, defined by $\Delta_c = \partial V/\partial S$ for a European Vanilla Call is given by

$$\Delta_c = N(d_1).$$

What does the delta of an option measure?

Denote the value of a call by C and the value of a put by P. By considering a portfolio which is long one asset, long one put and short on call, show that

$$C - P = S - Ee^{-r(T-t)}$$

and hence show that the delta Δ_p for a European Vanilla Put is

$$\Delta_p = N(d_1) - 1.$$

Give a financial reason why $\Delta_p < \Delta_c$.

Answer

(a) As advised, we assume that for a Euro Vanilla Call

$$V = SN(d_1) - Ee^{-r(T-t)}N(d_2).$$

Now

$$\frac{\partial}{\partial S}(N(d_1)) = N_{d_1} d_{1S}.$$

But

$$N(d_1) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{d_1} e^{-q^2/2}\, dq$$

$$\begin{aligned} \Rightarrow N_{d_1} &= \frac{1}{\sqrt{2\pi}} e^{-d_1^2/2} \\ \text{also } \mathrm{d_1S} &= \frac{1}{\sigma S\sqrt{T-t}}. \end{aligned}$$

Thus $(N(d_1))_S = \frac{e^{-d_1^2/2}}{(\sigma S\sqrt{2\pi(T-t)})}$

Similarly

$$\begin{aligned} (N(d_2))_S &= N_{d_2} d_2 S \\ &= \frac{1}{\sqrt{2\pi}} e^{-d_2^2/2} \times \frac{1}{\sigma S\sqrt{t-t}} \end{aligned}$$

So now

$$\begin{aligned} \Delta_c &= \frac{\partial V}{\partial S} = N(d_1) + \frac{e^{-d_1^2/2}}{\sigma\sqrt{2\pi(T-t)}} - \frac{Ee^{-r(T-t)}e^{-d_2^2/2}}{S\sigma\sqrt{2\pi(T-t)}} \\ &= N(d_1) + \frac{1}{\sigma\sqrt{2\pi(T-t)}} \left[e^{-d_1^2/2} - e^{-\log(S/E) - r(T-t) - d_2^2/2} \right] \end{aligned}$$

Now $d_1 - d_2 = \dfrac{\sigma^2(T-t)}{\sigma\sqrt{T-t}} = \sigma\sqrt{T-t}$

$$\begin{aligned} \Rightarrow d_1^2 &= d_2^2 + \sigma^2(T-t) + 2d_2\sigma\sqrt{T-t} \\ \Rightarrow \Delta &= N(d_1) + \frac{e^{-d_2^2/2}}{\sigma\sqrt{2\pi(T-t)}} \Big[e^{-\frac{\sigma^2}{2}(T-t) - d_2\sigma\sqrt{T-t}} \end{aligned}$$

$$
\begin{aligned}
& -e^{-\log(S/E)-r(T-t)}\Big] \\
= \;& N(d_1) \\
& +\frac{e^{-d_2^2/2}}{\sigma\sqrt{2\pi(T-t)}}\Big[e^{-\frac{\sigma^2}{2}(T-t)-\log(S/E)-r(T-t)+\frac{\sigma^2}{2}(T-t)} \\
& -e^{-\log(S/E)-r(T-t)}\Big] \\
= \;& N(d_1)
\end{aligned}
$$

The delta of an option measures the sensitivity of the option value to changes in the underlying and of course is the key hedging quantity.

(b) Now consider a portfolio $\Pi = S + P - C$. The payoff at expiry is

$$\rho = S + \max(\mathrm{E-S},0) - \max(\mathrm{S-E},0)$$

There are two cases:-

(i) $S \geq E \Rightarrow \rho = S + 0 - (S - E) \Rightarrow \rho = E$

(ii) $S/leE \Rightarrow \rho = S + E - S - 0 \Rightarrow \rho = E$

So arbitrage

$$\Rightarrow \Pi = Ee^{-r(T-t)}$$

$\Rightarrow$ Put/Call Parity $S + P - C = Ee^{-r(T-t)}$

i.e.

$$C - P = S - Ee^{-r(T-t)}$$

Thus $\Delta_c = \frac{\partial C}{\partial S}$, but $\frac{\partial C}{\partial S} - \frac{\partial P}{\partial S} = 1 - 0$, (from above).

$$
\begin{aligned}
\Rightarrow \Delta_P &= \Delta_C - 1 \\
\text{i.e. } \Delta_\mathrm{P} &= N)d_1) - 1
\end{aligned}
$$

The delta of a Put is less as a Call has an unlimited upside.