QUESTION

(a) Fine the vector equation and standard equation of the following planes:

(i) The plane Π_1 which passes through the point $P = (1, 2, 3)$ and has the normal vector $\mathbf{n} = 2\mathbf{i} - 1\mathbf{j} + 3\mathbf{k}$;

(ii) The plane Π_2 which passes through the three points $P_1 = (8, 8, -1),\ P_2 = (2, 0, -8),\ P_3 = (14, 30, 4)$;

(iii) The plane Π_3 which passes through the point $Q = (5, 0, -8)$ and is parallel to the plane with equation $6x - 3y + 9z = 11$.

(iv) Are any of the planes $\Pi_1,\ \Pi_2,\ \Pi_3$ parallel? Are any of them equal?

(b) Let $\mathbf{a} = 2\mathbf{i} + 2\mathbf{j} - \mathbf{k}$ and $\mathbf{b} = 2\mathbf{i} - \mathbf{j}$.

(i) Find the relation that must hold between x_1, x_2 and x_3 if the vector $\mathbf{x} = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}$ is to be written as $\mathbf{x} = s\mathbf{a} + t\mathbf{b}$ where s and t are scalars.

(ii) Show that the vector $\mathbf{c} = 2\mathbf{i} + 8\mathbf{j} - 3\mathbf{k}$ can be written as $\mathbf{c} = s\mathbf{a} + t\mathbf{b}$.

(iii) Fins s and t in this case.

ANSWER

(a) (i)

$$\mathbf{v}.\begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} . \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} = 9$$

$$2x - y + 3z = 9$$

(ii) $\mathbf{u} = \vec{P_2P_1} = 6\mathbf{i} + 8\mathbf{j} + 7\mathbf{k}$
$\mathbf{v} = \vec{p_2P_3} = 12\mathbf{i} + 30\mathbf{j} + 12\mathbf{k}$

$$\begin{aligned}
\mathbf{n} = \mathbf{u} \times \mathbf{v} &= \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 6 & 8 & 7 \\ 12 & 30 & 12 \end{vmatrix} \\
&= \begin{vmatrix} 8 & 7 \\ 30 & 12 \end{vmatrix} \mathbf{i} - \begin{vmatrix} 6 & 7 \\ 12 & 12 \end{vmatrix} \mathbf{j} + \begin{vmatrix} 6 & 8 \\ 12 & 30 \end{vmatrix} \mathbf{k} \\
&= (96 - 210)\mathbf{i} - (72 - 84)\mathbf{j} + (-180 - 96)\mathbf{k} \\
&= -114\mathbf{i} + 12\mathbf{j} + 84\mathbf{k}
\end{aligned}$$

This can be scaled to $-19\mathbf{i}+12\mathbf{j}+84\mathbf{k}$

$$\begin{aligned}
\mathbf{w}.\mathbf{n} = \vec{OP_2}.\mathbf{n} &= \begin{pmatrix} 2 \\ 0 \\ -8 \end{pmatrix} . \begin{pmatrix} -114 \\ 12 \\ 84 \end{pmatrix} \\
&= -228 - 672 = 900 \\
-114x + 12y + 84z &= 900
\end{aligned}$$

(iii) $\mathbf{u}.\mathbf{n} = \begin{pmatrix} 5 \\ 0 \\ -8 \end{pmatrix} .\mathbf{n}$ where $\mathbf{n} = 6\mathbf{i} - 3\mathbf{j} + 9\mathbf{k}$

$$6x - 3y + 9z = -42$$

(iv) To check for parallelism check normals: First and third are parallel. Since equations are inconsistent they are distinct.

(b) **(i)** $\mathbf{x}.(\mathbf{a}\times\mathbf{b}) = 0$

$$\mathbf{a}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 2 & -1 \\ 2 & -1 & 0 \end{vmatrix} = -\mathbf{i} - 2\mathbf{j} - 6\mathbf{k}$$

so $x_1 + 2x_2 + 6x_3 = 0$

(ii) $\mathbf{c}.(\mathbf{a}\times\mathbf{b}) = -2 - 16 + 2 + 18 = 0$

(iii) $\mathbf{c}\times\mathbf{a} = -t(\mathbf{a}\times\mathbf{b})$ and $\mathbf{c}\times\mathbf{b} = s(\mathbf{a}\times\mathbf{b})$

$$\mathbf{c}\times\mathbf{a} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{j} \\ 2 & 8 & -3 \\ 2 & 2 & -1 \end{vmatrix} = -2\mathbf{i} - 4\mathbf{j} - 12\mathbf{k}$$

so $t = -2$

$$\mathbf{c}\times\mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 8 & -3 \\ 2 & -1 & 0 \end{vmatrix} = -3\mathbf{i} - 6\mathbf{j} - 18\mathbf{k}$$

so $s = 3$