QUESTION

(a) Set up the following system of linear differential equations as a matrix proble:

$$\frac{dx}{dt} - x - 3y = 0,\ \frac{dy}{dt} - 3x - y - 4z = 0,\ \frac{dz}{dt} - 4y - z = 0.$$

(b) Write down the general form of the solution to the problem.

(c) Find the particular solution subject to the initial conditions $\frac{dx}{dt} = 6,\ \frac{dy}{dt} = -10,\ \frac{dz}{dt} = 8$ when $t = 0$.

ANSWER

(a)

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 & 3 & 0 \\ 3 & 1 & 4 \\ 0 & 4 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

(b) General solution $\mathbf{v}_1 e^{\lambda_1 t} + \mathbf{v}_2 e^{\lambda_2 t} + \mathbf{v}_3 e^{\lambda_3 t}$ where λ_i are vectors corresponding to λ_i.

$$\begin{vmatrix} 1-\lambda & 3 & 0 \\ 3 & 1-\lambda & 4 \\ 0 & 4 & 1-\lambda \end{vmatrix} = (1-\lambda)\left[(1-\lambda)^2 - 4^2\right] - 3\left[3(1-\lambda)\right]$$

$$= (1-\lambda)\left[(1-\lambda)^2 - 5^2\right]$$

$$= (1-\lambda)(-4-\lambda)(6-\lambda)$$

$\lambda = -4, 1, 6$

Eigenvectors:

$\lambda = -4$

$$\begin{pmatrix} 5 & 3 & 0 \\ 3 & 5 & 4 \\ 0 & 4 & 5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \Rightarrow k_1 \begin{pmatrix} -3 \\ 5 \\ -4 \end{pmatrix}$$

$\lambda = 1$

$$\begin{pmatrix} 0 & 3 & 0 \\ 3 & 0 & 4 \\ 0 & 4 & 0 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \Rightarrow k_2 \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix}$$

$\lambda = 6$

$$\begin{pmatrix} -5 & 3 & 0 \\ 3 & -5 & 4 \\ 0 & 4 & -5 \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \mathbf{0} \Rightarrow k_3 \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix}$$

General solution

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = k_1 \begin{pmatrix} -3 \\ 5 \\ -4 \end{pmatrix} e^{-4t} + k_2 \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} e^{t} + k_3 \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} e^{6t}$$

(c) For the particular solution solve

$$-4k_1 \begin{pmatrix} -3 \\ 5 \\ -4 \end{pmatrix} + k_2 \begin{pmatrix} 4 \\ 0 \\ -3 \end{pmatrix} + 6k_3 \begin{pmatrix} 3 \\ 5 \\ 4 \end{pmatrix} = \begin{pmatrix} 6 \\ -10 \\ 8 \end{pmatrix}$$

to get

$$\begin{aligned} k_1 &= -\frac{1}{2} \\ k_2 &= 0 \\ k_3 &= 0 \end{aligned}$$