Question
Derive the components of

(a) velocity and acceleration in cylindrical polar coordinates,

(b) velocity in spherical polar coordinates.

The expression for acceleration in spherical polar coordinates is

$$\begin{aligned}\mathbf{a} &= (\ddot{r} - r\dot{\theta}^2 - r\sin^2\theta\dot{\phi}^2)\mathbf{r}_r + \left[\frac{1}{r}\frac{d}{dt}(r^2\dot{\theta}) - r\sin\theta\cos\theta\dot{\phi}^2\right]\mathbf{e}_\theta \\ &\quad + \frac{1}{r\sin\theta}\frac{d}{dt}(r^2\sin^2\theta\dot{\phi})\mathbf{e}_\phi\end{aligned}$$

Answer

(a)

$$\begin{aligned}\mathbf{r} &= r\mathbf{e}_r + z\mathbf{e}_z \\ \mathbf{v} = \dot{\mathbf{r}} &= \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r + \dot{z}\mathbf{e}_z + z\dot{\mathbf{e}}_z \\ \text{Now } \dot{\mathbf{e}}_z &= 0 \text{ as in a fixed direction} \\ \mathbf{e}_r &= \mathbf{i}\cos\theta + \mathbf{j}\sin\theta \\ \dot{\mathbf{e}}_r &= \dot{\theta}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = \dot{\theta}\mathbf{e}_\theta \\ \text{So } \mathbf{v} &= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + \dot{z}\mathbf{e}_z \\ \mathbf{a} = \dot{\mathbf{v}} &= \ddot{r}\mathbf{e}_r + \dot{r}\dot{\mathbf{e}}_r + (\dot{r\dot{\theta}})\mathbf{e}_\theta + r\dot{\theta}\dot{\mathbf{e}}_\theta + \ddot{z}\mathbf{e}_z \\ \text{Now } \dot{\mathbf{e}}_\theta &= \frac{d}{dt}(-\sin\theta\mathbf{i} + \cos\theta\mathbf{j}) = -\dot{\theta}\mathbf{e}_r \\ \text{So } \mathbf{a} &= \ddot{r}\mathbf{e}_r + \dot{r}\dot{\theta}\mathbf{e}_\theta + (\dot{r\dot{\theta}})\mathbf{e}_\theta - r\dot{\theta}^2\mathbf{e}_r + \ddot{z}\mathbf{e}_z \\ \Rightarrow \mathbf{a} &= (\ddot{r} - r\dot{\theta}^2)\mathbf{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\mathbf{e}_\theta + \ddot{z}\mathbf{e}_z\end{aligned}$$

(b)

$$\begin{aligned} & \mathbf{r} = r\mathbf{e}_r \quad \mathbf{v} = \dot{r}\mathbf{e}_r + r\dot{\mathbf{e}}_r \\ \text{Now } \mathbf{e}_r &= \mathbf{i}\sin\theta\cos\phi + \mathbf{j}\sin\theta\sin\phi + \mathbf{k}\cos\theta \\ \dot{\mathbf{e}}_r &= \dot{\theta}(\mathbf{i}\cos\theta\cos\phi + \mathbf{j}\cos\theta\sin\phi - \mathbf{k}\sin\theta) \\ &\quad + \dot{\phi}(-\mathbf{i}\sin\theta\sin\phi + \mathbf{j}\sin\theta\cos\phi) \\ &= \dot{\theta}\mathbf{e}_\theta + \dot{\phi}\sin\theta\mathbf{e}_\phi \\ \text{So } \mathbf{v} &= \dot{r}\mathbf{e}_r + r\dot{\theta}\mathbf{e}_\theta + r\dot{\phi}\sin\theta\mathbf{e}_\phi\end{aligned}$$