QUESTION

Show that for a Binomial distribution with parameters n and p

$$p(x) = \frac{n+1-x}{x}\frac{p}{q}p(x-1)$$

(i) For $n = 5$ and $p = 0.4$ use this recurrence relation together with the value of $p(0)$ to evaluate all the values of $p(x)$

(ii) Use the expression to prove that the mode of the distribution is the integer value of $(n+1)p$ and the distribution has two modes $(n+1)p$ and $(n+1)p-1$ if $(n+1)p$ is an integer.

ANSWER

$$p(x) = \left(\begin{array}{c} n \\ x \end{array}\right) p^x q^{n-x}$$

$$p(x-1) = \frac{n!}{(x-1)!(n+1-x)!}p^{x-1}q^{n+1-x}$$

$$\begin{aligned} \frac{p(x)}{p(x-1)} &= \frac{n!(x-1)!(n+1-x)!}{x!(n-x)!n!}\frac{p^{x-1}q^{n-x}}{p^{x-1}q^{n+1-x}} \\ &= \frac{n+1-x}{x}\frac{p}{q} \end{aligned}$$

(i) $p(0) = 0.6^5 = 0.07776$

$p(1) = \frac{5+1-1}{1}\frac{0.4}{0.6}p(0) = 0.2592$
$p(2) = \frac{5+1-2}{2}\frac{0.4}{0.6}p(1) = 0.3456$
$p(3) = \frac{5+1-3}{3}\frac{0.4}{0.6}p(2) = 0.2304$
$p(4) = \frac{5+1-4}{4}\frac{0.4}{0.6}p(3) = 0.0768$
$p(5) = \frac{5+1-5}{5}\frac{0.4}{0.6}p(4) = 0.01024$

(ii)

$$p(x) \geq p(x-1) \text{ if } \frac{n+1-x}{x}\frac{p}{1-p} \geq 1$$

$$\begin{aligned} (n+1-x)p &\geq x(1-p) \\ (n+1)p &\geq x \end{aligned}$$

Hence $p(x)$ increases if $x \leq (n+1)p$. Otherwise it decreases. Hence the mode is the integer value of $(n+1)p$. If this is an integer we say $p(k) = p(k-1)$, hence there are two modes $(n+1)p$ and $(n+1)p-1$.