Question

For each of the following Möbius transformations, determine the fixed points, normalize so that the determinant is 1, and determine the type (parabolic, elliptic, loxodromic) of the transformation. If the transformation is elliptic or loxodromic, determine its multiplier.

(a) $m(z) = \frac{3z-3}{z+2}$;

(b) $m(z) = \frac{-z-16}{2z+2}$;

(c) $m(z) = 8z + 3 + i$;

(d) $m(z) = \frac{4}{7z+2i}$;

Answer

(a) $m(z) = \dfrac{3z-3}{z+2}$, $\det(m) = 9$, so m <u>normalized</u> is $m(z) = \dfrac{z-1}{\frac{1}{3}z + \frac{2}{3}}$ (divide all coefficients by $\sqrt{\det(\mathrm{m})} = 3$)

<u>fixed points</u>:

$$z = \frac{z-1}{\frac{1}{3}z + \frac{2}{3}}$$

$$\Rightarrow \frac{1}{3}z^2 + \frac{2}{3}z = z - 1$$

$$\Rightarrow \frac{1}{3}z^2 - \frac{1}{3}z + 1 = 0$$

$$\Rightarrow z^2 - z + 3 = 0$$

$$\Rightarrow z = \frac{1}{2}(1 \pm \sqrt{1-12})$$

$$\Rightarrow \underline{z = \frac{1}{2}(1 \pm \sqrt{11}i)}$$

$\tau(m) = \left(1 + \dfrac{2}{3}\right)^2 = \dfrac{25}{9} < 4$ and so <u>m is elliptic</u>.

multiplier is λ^2, where $(\lambda+\lambda^{-1})^2 = \tau(m) = \dfrac{25}{9}$

$$\begin{aligned} \lambda^2 + 2 + \lambda^{-2} &= \frac{25}{9} \\ \lambda^2 - \frac{7}{9} + \lambda^{-2} &= 0 \\ \lambda^4 - \frac{7}{9}\lambda^2 + 1 &= 0 \end{aligned}$$

$$\underline{\lambda^2 = \frac{1}{2}\left(\frac{7}{9} \pm \sqrt{\left(-\frac{7}{9}\right)^2 - 4}\right)}$$

(Can choose either.)

(b) $m(z) = \dfrac{-z-16}{2z+2}$, $\det(m) = 30$, so m normalized is

$$m(z) = \frac{\frac{-1}{\sqrt{30}}z - \frac{16}{\sqrt{30}}}{\frac{2}{\sqrt{30}}z + \frac{2}{\sqrt{30}}}$$

fixed points:

$z = \dfrac{-z-16}{2z+2}$

$\Rightarrow 2z^2 + 2z = -z - 16$

$\Rightarrow 2z^2 + 3z + 16 = 0$

$\Rightarrow z = \dfrac{1}{4}(-3 \pm \sqrt{9-128})$

$\Rightarrow \underline{z = \dfrac{1}{4}(-3 \pm \sqrt{119}i)}$

$\tau(m) = \left(-\dfrac{1}{\sqrt{30}} + \dfrac{2}{\sqrt{30}}\right)^2 = \dfrac{1}{30} < 4$ and so m is elliptic.

multiplier is λ^2, where $(\lambda+\lambda^{-1})^2 = \tau(m) = \dfrac{1}{30}$

$$\begin{aligned} \lambda^2 + 2 + \lambda^{-2} &= \tfrac{1}{30} \\ \lambda^2 + \tfrac{59}{30} + \lambda^{-2} &= 0 \\ \lambda^4 + \tfrac{59}{30}\lambda^2 + 1 &= 0 \end{aligned}$$

$$\underline{\lambda^2 = \frac{1}{2}\left(-\frac{59}{30} \pm \sqrt{\left(\frac{59}{30}\right)^2 - 4}\right)}$$

(c) $m(z) = 8z + 3 + i$, $\det(m) = 8$, so m <u>normalized</u> is $m(z) = \dfrac{\sqrt{8}z + \frac{3+i}{\sqrt{8}}}{0z + \frac{1}{\sqrt{8}}}$

<u>fixed points</u>: one is $\underline{\infty}$. The other is the solution of $z = 8z + 3 + i \Rightarrow$ $7z = -3 - i \Rightarrow z = \dfrac{-3-i}{7}$

$$\tau(m) = \left(\sqrt{8} + \frac{1}{\sqrt{8}}\right)^2$$

$= 8 + 2 + \dfrac{1}{8} = \dfrac{81}{8} > 4$ and so $\underline{m \text{ is loxodromic}}$.

<u>multiplier is λ^2</u>, where $(\lambda + \lambda^{-1})^2 = \tau(m) = \dfrac{81}{8}$

$$\begin{aligned} \lambda^2 + 2 + \lambda^{-2} &= \frac{81}{8} \\ \lambda^2 + \frac{-65}{8} + \lambda^{-2} &= 0 \\ \lambda^4 - \frac{65}{8}\lambda^2 + 1 &= 0 \end{aligned}$$

$$\underline{\lambda^2 = \frac{1}{2}\left(\frac{65}{8} \pm \sqrt{\left(\frac{-65}{8}\right)^2 - 4}\right) = 8 \text{ or } \frac{1}{8}}$$

(d) $m(z) = \dfrac{4}{7z + 2i} = \dfrac{0z + 4}{7z + 2i}$, $\det(m) = -28$, so m <u>normalized</u> is

$$m(z) = \frac{\frac{4i}{\sqrt{28}}}{\frac{7i}{\sqrt{28}}z + \frac{-2}{\sqrt{28}}}$$

<u>fixed points</u>:

$z = \dfrac{4}{7z + 2i}$

$\Rightarrow 7z^2 + 2iz = 4$

$\Rightarrow 7z^2 + 2iz - 4 = 0$

$\Rightarrow z = \dfrac{1}{14}(-2i \pm \sqrt{-4 + 112})$

$\Rightarrow \underline{z = \dfrac{1}{14}(-2i \pm \sqrt{108})}$

$\tau(m) = \left(-\dfrac{2}{\sqrt{28}}\right)^2 = \dfrac{4}{28} = \dfrac{1}{7} < 4$ and so $\underline{m \text{ is elliptic}}$.

multiplier is λ^2, where $(\lambda + \lambda^{-1})^2 = \tau(m) = \frac{1}{7}$

$$\begin{aligned} \lambda^2 + 2 + \lambda^{-2} &= \frac{1}{7} \\ \lambda^2 + \frac{13}{7} + \lambda^{-2} &= 0 \\ \lambda^4 + \frac{13}{7}\lambda^2 + 1 &= 0 \end{aligned}$$

$$\underline{\lambda^2 = \frac{1}{2}\left(-\frac{13}{7} \pm \sqrt{\left(\frac{13}{7}\right)^2 - 4}\right)}$$