

Question

Let $T = \{2 + i, 4, -2 + 3i\}$. There are six Möbius transformations m satisfying $m(T) = T$. Find explicit expressions for two of them (other than the identity).

Answer

$m(2 + i) = 0$, $m(4) = \infty$, $m(-2 + 3i) = 1$:

$$\begin{aligned} m(z) &= \frac{z - (2 + i)}{z - 4} \cdot \frac{-2 + 3i - 4}{-2 + 3i - (2 + i)} \\ &= \frac{z - (2 + i)}{z - 4} \cdot \frac{-6 + 3i}{-4 + 2i} \\ &= \frac{(-6 + 3i)z + 15}{(-4 + 2i)z + (16 - 8i)}. \end{aligned}$$

$J(z) = \frac{1}{z}$ permutes $\{0, 1, \infty\}$ as does $p(z) = -z + 1$ and so mJm^{-1}, mpm^{-1} permute $\{2 + i, 4, -2 + 3i\}$.