Question

Write down an explicit Möbius transformation taking the circle $\{z \in \mathbf{C} \,|\, |z - 2| = 3\}$ to the circle $\{z \in \mathbf{C} \,|\, \mathrm{Re}(z) + \mathrm{Im}(z) = 1\}$.

Answer

Take 3 points on $A = \{|z-2| = 3\}$, say $z_1 = 2+3i, z_2 = -1, z_3 = 5$, and 3 points on $B = \{\mathrm{Re(z)} + \mathrm{Im(z)} = 1\}$, say $w_1 = 1, w_2 = i$, and $w_3 = -2+3i$. Now find $m \in \mathrm{Möb}^+$ with $m(z_k) = w_k,\ \ k = 1,2,3$. Start by finding $n(z)$ with $n(2+3i) = 0, n(-1) = \infty$ and $n(5) = 1$, and $p(z)$ with $P(1) = 0, p(i) = \infty$ and $p(-2+3i) = 1$ and then set $m = p^{-1} \cdot n$.

$$
\begin{aligned}
n(z) &= \frac{z-(2+3i)}{z+1} \cdot \frac{6}{5-(2+3i)} \\
&= \frac{6}{3-3i} \cdot \frac{z-(2+3i)}{z+1} \\
&= \frac{6z - 6(2+3i)}{(3-3i)z + (3-3i)}
\end{aligned}
$$

$$
\begin{aligned}
p(z) &= \frac{z-1}{z-i} \cdot \frac{-2+3i-i}{-2+3i-1} \\
&= \frac{z-1}{z-i} \cdot \frac{-2+2i}{-3+3i} \\
&= \frac{2(z-1)}{3(z-i)} = \frac{2z-2}{3z-3i}
\end{aligned}
$$

$$
p^{-1}(z) = \frac{-3iz+2}{-3z+2}
$$

$$
\begin{aligned}
m(z) &= p^{-1}(n(z)) \\
&= \frac{-3in(z)+2}{-3n(z)+2} \\
&= \frac{-3i(6z - 6(2+3i)) + 2((3-3i)z + (3-3i))}{-3(6z - 6(2+3i)) + 2((3-3i)z + (3-3i))} \\
&= \frac{(6-24i)z - 48 + 30i}{(-12-6i)z + 42 + 48i}
\end{aligned}
$$

(There are many others.)