Question
In the proof that the group of Möbius transformations acts transitively on the set of circles in $\overline{\mathbf{C}}$, we use the fact that three distinct points in $\overline{\mathbf{C}}$ determine a circle in $\overline{\mathbf{C}}$.

Prove this fact in a single special case: determine the center and radius of the circle determined by $2+i$, $3-i$, and $-7i$.

Answer
The circle in $\mathbf{C}$ determined by $2+i, 3-i, -7i$ (first note that these 3 points do not lie on a line in $\mathbf{C}$, sine the line through $2+i$ and $3-i$ intersects the imaginary axis at $5i \neq -7i$).

- The midpoint of the line segment through $2+i$ and $3-i$ is $\frac{1}{2}(2+i+3-i) = \frac{5}{2}$ and its slope is $\frac{-1-1}{3-2} = -2$.

  The perpendicular bisector then has equation $y - 0 = +\frac{1}{2}\left(x - \frac{5}{2}\right)$

  $\Rightarrow \underline{y = \frac{1}{2}x - \frac{5}{4}}$

- The midpoint of the line segment through $2+i$ and $-7i$ is $\frac{1}{2}(2+i-7i) = 1-3i$ and its slope is $\frac{-7-1}{0-2} = 4$.

  The perpendicular bisector then has equation $y + 3 = -\frac{1}{4}(x-1) = -\frac{1}{4}x + \frac{1}{4}$

  $\Rightarrow \underline{y = -\frac{1}{4}x - \frac{11}{4}}$

These perpendiculars are both diameters of the desired circle and hence intersect at the center:

$$\begin{aligned} \frac{1}{2}x - \frac{5}{4} &= -\frac{1}{4}x - \frac{11}{4} \\ \frac{3}{4}x &= -\frac{6}{4} \end{aligned}$$

$\underline{x = -2}$
$\underline{y = \frac{1}{2}x - \frac{5}{4} = -\frac{9}{4}}$

So, the center of the circle is $\underline{-2 - \frac{9}{4}i = a}$.

<u>Its radius is</u>:

$$\begin{aligned}
|-2 - \tfrac{9}{4}i - (-7i)| &= |-2 + \tfrac{19}{4}i| = 5.154 \\
|-2 - \tfrac{9}{4}i - (2+i)| &= |-4 - \tfrac{13}{4}i| = 5.154 \\
|-2 - \tfrac{9}{4}i - (3-i)| &= |-5 - \tfrac{5}{4}i\,| = 5.154
\end{aligned}$$