

Question

Let A be the circle in \mathbf{C} with center $a = 2 + 2i$ and radius $r = 1$. Let $m(z) = 1 + \frac{1}{z}$. Find the equation of the circle $m(A)$.

Answer

Start by putting the equation for A into standard form:

$$\begin{aligned} |z - (2 + 2i)|^2 &= 1 \\ (z - (2 + 2i))(\bar{z} - (2 - 2i)) &= 1 \\ z\bar{z} - (2 - 2i)z - (2 + 2i)\bar{z} + |2 + 2i|^2 &= 1 \\ \underline{z\bar{z} - (2 - 2i)z - (2 + 2i)\bar{z} + 7} &= 0 \end{aligned}$$

$$m(z) = 1 + \frac{1}{z} = \omega$$

$$\Rightarrow \frac{1}{z} = \omega - 1 \Rightarrow z = \frac{1}{\omega - 1} \text{ plug into the equation for } A\text{:}$$

$$\left(\frac{1}{\omega - 1}\right) \left(\frac{1}{\bar{\omega} - 1}\right) - (2 - 2i)\frac{1}{\omega - 1} - (2 + 2i)\frac{1}{\bar{\omega} - 1} + 7 = 0 \text{ clear denominators:}$$

$$1 - (2 - 2i)(\bar{\omega} - 1) - (2 + 2i)(\omega - 1) + 7(\omega - 1)(\bar{\omega} - 1) = 0 \text{ and simplify:}$$

$$1 - (2 - 2i)\bar{\omega} + 2 - 2i - (2 + 2i)\omega + 2 + 2i + 7\omega\bar{\omega} - 7\bar{\omega} - 7\omega + 7 = 0$$

$$\boxed{7\omega\bar{\omega} + (-2 - 2i - 7)\omega + (-2 + 2i - 7)\bar{\omega} + 12 = 0}$$

which is the equation for $m(A)$. $7\omega\bar{\omega} + (-9 - 2i)\omega + (-9 + 2i)\bar{\omega} + 12 = 0$

$$\left|\omega - \left(\frac{9 - 2i}{7}\right)\right|^2 = \frac{1}{49}$$