QUESTION

Use the Fourier transform to solve the heat equation

$$f_t = f_{xx},\ t > 0,\ -\infty < x < \infty$$

given that

$$f(x,0) = e^{-\frac{x^2}{2}},\ f, f_x \to 0 \text{ as } x \to \pm\infty$$

.

ANSWER

$f_t = f_{xx},\ t > 0,\ -\infty < x < \infty,\ f(x,0) = e^{-\frac{x^2}{2}},\ f, f_x \to 0$ as $|x| \to 0.$

Fourier transform with respect to x:

$F_t = -k^2 F$

$F(t,k) = F(0,k)e^{-k^2 t}$

$F(0,k) = e^{-\frac{k^2}{2}}$

$F(t,k) = e^{-\frac{k^2(1+2t)}{2}}$

Inverse Fourier transform

$f(t,x) = \dfrac{1}{2\pi}\displaystyle\int_{-\infty}^{\infty} e^{-\frac{k^2(1+2t)}{2} + ikx}\, dk$

Change of variable

$\overline{k} = \sqrt{1+2t}k$ and $\overline{x} = \frac{x}{\sqrt{1+2t}}$

$$\begin{aligned} f(t,x) &= \frac{1}{2\pi}\frac{1}{\sqrt{1+2t}}\int_{-\infty}^{\infty} e^{-\frac{\overline{k}^2}{2} + i\overline{k}\overline{x}}\, d\overline{k} \\ &= \frac{1}{\sqrt{1+2t}} e^{-\frac{\overline{x}^2}{2}} \text{ by the result we have used before} \\ &= \frac{1}{\sqrt{1+2t}} e^{-\frac{x^2}{2(1+2t)}} \end{aligned}$$