QUESTION

Find the function whose Fourier transform is

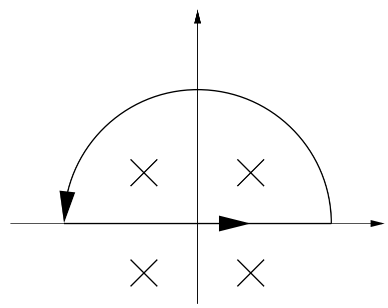
$$\frac{1}{16+\xi^4}$$

by integrating $\dfrac{e^{izx}}{(16+z^4)}$ around a large semicircle.

ANSWER

$$F(\xi)=\frac{1}{16+\xi^4}; \quad f(x)=\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{e^{izx}}{16+z^4}\,dz$$

For $x>0$ we choose in the upper half plane:



Poles at $z=2e^{\frac{i\pi}{4}}=\sqrt{2}(1+i)$ and $z=2e^{\frac{3i\pi}{4}}=\sqrt{2}(-1+i)$
(and two more poles in the lower half plane).
$\text{Res}\left(\dfrac{e^{izx}}{16+z^4},z_0\right)=\dfrac{e^{iz_0x}}{4z_0^3}=\dfrac{z_0e^{iz_0x}}{-64}$

$$\begin{aligned}
f(x) &= \frac{1}{2\pi}2\pi i\left(-\frac{1}{32}e^{\frac{i\pi}{4}}e^{i\sqrt{2}(1+i)x}-\frac{1}{32}e^{\frac{3i\pi}{4}}e^{i\sqrt{2}(-1+i)x}\right)\\
&= -\frac{i}{32}e^{-\sqrt{2}x}\left(e^{i\sqrt{2}x+\frac{i\pi}{4}}-e^{-i\sqrt{2}x-\frac{i\pi}{4}}\right)\\
&= -\frac{i}{32}e^{-\sqrt{2}x}\left(e^{i\sqrt{2}x+\frac{i\pi}{4}}+e^{-i\sqrt{2}x+\frac{3i\pi}{4}}\right)\\
&= \frac{1}{16}e^{-\sqrt{2}x}\sin\left(\sqrt{2}x+\frac{\pi}{4}\right),\ x>0
\end{aligned}$$

From the expression for $f(x), f(-x)=\overline{f(x)}=f(x)$, so for any x,

$$f(x)=\frac{1}{16}e^{-\sqrt{2}|x|}\sin\left(\sqrt{2}|x|+\frac{\pi}{4}\right)$$