QUESTION

(a) Sketch the locus in the ω-plane of the point

$$\omega = \frac{1-z}{z(1+z)(2+z)}$$

as z moves down the imaginary axis with an indentation to the right to avoid 0.

(b) Hence by using Nyquist theory find the values of $k > 0$ for which the closed loop system with

$$A(s) = \frac{1-s}{(1+s)(1+2s)}, \quad B(s) = \frac{k}{s}$$

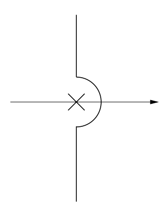
is stable.

(c) Check your answer by looking for the poles of the transfer function $f = \frac{A}{(1+AB)}$ in the right half plane directly.

ANSWER

(a)

$$w = \frac{1-z}{z(1+z)(2+z)}$$



Parameterise $z = iy,\ \infty > y \geq \varepsilon$ and $-\varepsilon \geq y > -\infty$

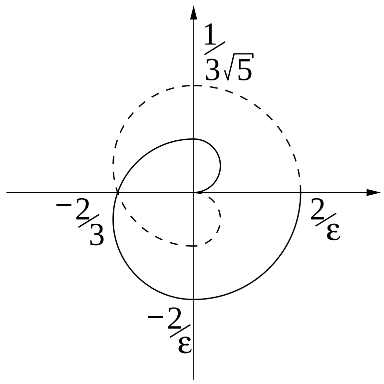
$$\begin{aligned} w &= \frac{(1-iy)(1-iy)(2-iy)}{iy(1+y^2)(4+y^2)} \\ &= -i\frac{(1-y^2-2iy)(2-iy)}{y(1+y^2)(4+y^2)} \\ &= \frac{-i(2-2y^2-2y^2-4iy-iy+iy^3)}{y(1+y^2)(4+y^2)} \\ &= i\frac{4y^2-2}{y(1+y^2)(4+y^2)} + \frac{y^2-5}{(1+y^2)(4+y^2)} \end{aligned}$$

As $|y| \to \infty, w \approx 4y^{-3}i + y^{-2}$

At $y = \pm\sqrt{5}$, Re$w = 0$, Im$w = \pm\frac{1}{3\sqrt{5}}$

At $y = \pm\frac{1}{\sqrt{2}}$, Im$w = 0$, Re$w = -\frac{2}{3}$ As $|y| \to 0$, $w \approx -\frac{i}{2}y^{-1} - \frac{5}{4}$

Parameterise $z = \varepsilon e^{i\theta}$, $\frac{\pi}{2} \geq \theta \geq -\frac{\pi}{2}$
$w \approx \frac{1}{2z} = \frac{1}{2\varepsilon}e^{-i\theta}$



(b) $A(s) = \frac{1-s}{(1+s)(1+2s)}$ $B(s) = \frac{k}{s}$ $h(S) = A(s)B(s) = k(s)$ The loop system is stable $\Leftrightarrow$ $h(s)$ does not wind around $-1 \Leftrightarrow w(s)$ does not wind around $-\frac{1}{k} \Leftrightarrow -\frac{1}{k} < -\frac{2}{3} \Leftrightarrow \frac{1}{k} > \frac{2}{3} \Leftrightarrow k < \frac{3}{2}$(assuming $k > 0$)

(c) The system is stable if the transfer function

$$f(s) = \frac{A(s)}{1 + A(s)B(s)} = \frac{s(1-s)}{s(1+s)(2+s) + (1-s)k}$$

has two poles in the right half plane.

Poles of $f(s)$ are zeros of $s(1+s)(2+s) + (1-s)k$. Let us find the marginal value of k that gives $f(s)$ a pole just on the imaginary axis, i.e. $s = iy$.

We solve $iy(1+iy)(2+iy) + (1-iy)k = 0$ for y and k.

i.e., $iy(1 - y^2 - k) + (k - 3y^2) = 0$

The solution $y = 0$, $k = 0$ is not the one we are after. The other solution is $y^2 = \frac{1}{2}$, $k = \frac{3}{2}$ as in 1(b). One would still have to show that Re$(s) > 0$ for $k > \frac{3}{2}$.