Question

A thin wire is stretched along the x-axis and is heated by an electric current. The temperature T of the wire varies with x and satisfies the differential equation:

$$\frac{d^2T}{dx^2} - p^2T = -k,$$

where p and k are nonzero constants. Put $T = \frac{k}{p^2} + y$ and show that y satisfies the differential equation:

$$\frac{d^2y}{dx^2} - p^2y = 0.$$

Hence show that the general solution of the equation for T is:

$$T = \frac{k}{p^2} + Ae^{px} + Be^{-px},$$

where A and B are constants. Find the values of A nad B given that Y satisfies the conditions $T = 0$ at $x = a$ and also $T = 0$ at $x = -a$ (where $a \neq 0$), and show that in this case

$$T = \frac{k}{p^2}\left(1 - \frac{\cosh px}{\cosh pa}\right)$$

Answer

$\frac{d^2T}{dx^2} - p^2T = -k$ (*)

If $T = \frac{k}{p^2} + y$, then (since $\frac{k}{p^2}$ is a constant)

$\rightarrow \frac{dT}{dx} = \frac{dy}{dx}$ and $\frac{d^2T}{dx^2} = d^y dx^2$

Substituting into (*) gives $\frac{d^2y}{dx^2} - p^2\left(\frac{k}{p^2} + y\right) = -k$

and so $\frac{d^2y}{dx^2} - p^2y = 0$ (**)

auxiliary equation for (**) $\lambda^2 - p^2 = 0$

which has two distinct real roots: $\lambda_1, \lambda_2 = \pm p$

General solution for (**): $y = Ae^{px} + Be^{-px}$

Hence the general solution for (*) is given by

$T = \frac{k}{p^2} + y = \frac{k}{p^2} + Ae^{px} + Be^{-px}$

Boundary conditions:

$T(a) = 0$ gives $= \frac{k}{p^2} + Ae^{pa} + Be^{-pa}$ (1)

$T(-a) = 0$ gives $= \frac{k}{p^2} + Ae^{-pa} + Be^{pa}$ (2)

Equation (1) - Equation(2) $A(e^{pa} - e^{-pa)) + B(e^{-pa} - e^{pa}} = 0$

and so $A(e^{pa} - e^{-pa}) = B(e^{pa} - e^{-pa})$

from which we deduce that $A = B$ (note $pa \neq 0$ by assumption, and so $(e^{pa} - e^{-pa}) \neq 0$)

Therefore $T = \frac{k}{p^2} + y = \frac{k}{p^2} + A(e^{px} + e^{-px}) = \frac{k}{p^2} + 2A\cosh(px)$
Boundary condition: $T(a) = 0$ gives
$0 = \frac{k}{p^2} + 2A\cosh(pa) \Rightarrow A = \frac{A}{2p^2\cosh(pa)}$
Hence the particular solution:

$$T = \frac{k}{p^2}\left(1 - \frac{\cosh(px)}{\cosh(pa)}\right)$$