Question

Solve the following differential equations, using the specified boundary conditions.

(a) $\frac{d^2y}{dx^2} + 3\frac{dy}{dx} + 2y = 0$, where $y = 2$ and $\frac{dy}{dx} = -3$ when $x = 0$

(b) $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 13y = 0$, where $y = 1$ and $\frac{dy}{dx} = 2$ when $x = 0$

(c) $\frac{d^2y}{dx^2} - 6\frac{dy}{dx} + 9y = 0$, where $y = 2$ and $\frac{dy}{dx} = 4$ when $x = 0$

Answer

(a) Auxiliary equation: $\lambda^2 + 3\lambda + 2 = 0$

Factorises to give: $(\lambda + 1)(\lambda + 2) = 0$

so two distinct real roots $\lambda_1 = -1$ and $\lambda_2 = -2$

$$\begin{aligned}
\text{General solution} : y &= Ae_{-x} + Be^{-2x} \\
\frac{dy}{dx} &= -Ae^{-x} - 2e^{-2x} \\
\text{Boundary conditions} : 2 &= A + B \\
-3 &= -A - 2B \\
\Rightarrow B &= 1 \text{ and } A = 1 \\
\text{Particular solution} : y &= e^{-x} + Be^{-2x}
\end{aligned}$$

(b) Auxiliary equation: $\lambda^2 + 6\lambda + 13 = 0$

$\lambda_1, \lambda_2 = \frac{-6 \pm \sqrt{36-52}}{2}$

(a pair of complex conjugate roots)

$$\begin{aligned}
\text{General solution} : y &= e_{-3x}(A\cos(2x) + B\sin(2x)) \\
\frac{dy}{dx} &= e^{-3x}((2B - 2A)\cos(2x) - (2A + 3B)\sin 2x) \\
\text{Boundary conditions} : 1 &= A \\
2 &= 2B - 3A \\
\Rightarrow a &= 1 \text{ and } B = \frac{5}{2} \\
\text{Particular solution} : y &= e^{-3x}(\cos 2x + \frac{5}{2}\sin 2x)
\end{aligned}$$

(c) Auxiliary equation: $\lambda^2 - 6\lambda + 9 = 0$

Factorises to give: $(\lambda + 3)^2 = 0$

so two repeated real roots $\lambda_1 = \lambda_2 = 3$

$$\begin{aligned}
\text{General solution : } y &= (A + Bx)e^{3x} \\
\frac{dy}{dx} &= 3Ae^{3x} + Be^{3x} + 3Bxe^{3x} \\
\text{Boundary conditions : } 2 &= A \\
4 &= 3A + B \\
\Rightarrow A &= 2 \text{ and } B = -2 \\
\text{Particular solution : } y &= 2(1 - x)e^{3x}
\end{aligned}$$