Question
If $yu_x + u_y = 0$ for all x, y and $u(x,0) = exp(-x)$ for all x, find $u(x,y)$.

Answer
From Lagrange's equations,
$$\frac{dx}{d\xi} = y, \ \frac{dy}{d\xi} = 1, \ \frac{du}{d\xi} = 0$$
$$\Rightarrow \frac{dx}{dy} = \frac{y}{1}, \ \frac{du}{d\xi} = 0$$
$$\Rightarrow x = \frac{y^2}{2} + b, \ u = a \ (b, \ a \ consts)$$
$$\Rightarrow$$
$$(\star) \begin{cases} x - \dfrac{y^2}{2} & = \ const \\ u & = \ cpmst \end{cases}$$
$$\Rightarrow u = f\left(x - \frac{y^2}{2}\right)$$

If $u = e^{-x}$ at $y = 0$, then
$e^{-x} = f(x)$ i.e., $f(\eta) = e^{-\eta}$ for any η.
$$\Rightarrow \underline{u = e^{-(x^2 - \frac{y^2}{2})}}$$