Question

Suppose that the joint pdf of X and Y is as follows:

$$f(x,y) = \begin{cases} \frac{15}{4}x^2 & \text{if } 0 \le y \le 1-x^2 \\ 0 & \text{otherwise} \end{cases}$$

(a) Determine the marginal pdf's of X and Y. Are X and Y independent?

(b) Find the conditional pdf of Y given that $X = x$.

(c) Find $E(Y|X = x)$.

Answer

$f(x,y) = \dfrac{15}{4}x^2, \quad 0 \le y \le 1-x^2$

(a) First check that its a pdf

$$\begin{aligned} & \int\int_{0\le y\le 1-x^2} \frac{15}{4}x^2\,dy\,dx \\ = & \frac{15}{4}\int_{-1}^{1} x^2\,dx \int_0^{1-x^2} dy \\ = & \frac{15}{4}\int_{-1}^{1} x^2(1-x^2)\,dx = 1 \end{aligned}$$

$f_X(x) = \dfrac{15}{4}x^2(1-x^2), \quad -1 \le x \le 1$

$f_Y(y) = \dfrac{15}{4}\displaystyle\int_{-\sqrt{1-y}}^{\sqrt{1-y}} x^2\,dx = \dfrac{5}{2}(1-y)^{\frac{3}{2}}, \quad 0 < y < 1$

Since

$$\begin{aligned} & 1 - x^2 \ge y \\ \Rightarrow\ & x^2 \le 1-y \\ \Rightarrow\ & -\sqrt{1-y} \le x \le \sqrt{1-y} \end{aligned}$$

X and Y are not independent.

(b)

$$\begin{aligned} f(y|X=x) = \frac{f(x,y)}{f_X(x)} &= \frac{15}{4}x^2 \cdot \frac{1}{\frac{15}{4}x^2(1-x^2)} \\ &= \frac{1}{1-x^2}, \quad 0 \le y \le 1-x^2. \end{aligned}$$

$Y|X = x \sim \text{Uniform}(0, 1-x^2)$ where $\ -1 \le x \le 1$

(c) $E(Y|X = x) = \dfrac{1 - x^2}{2}$ where $-1 \leq x \leq 1$.