Question
A pdf is defined by

$$f(x,y) = \begin{cases} C(x+2y) & \text{if } 0 < x < 2 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the value of C.

(b) Find the marginal pdf's of X and Y. Are X and Y independent?

(c) Find the pdf of the random variable $Z = \dfrac{9}{(X+1)^2}$.

Answer
$f(x,y) = C(x+2y) \quad 0 < x < 2,\ 0 < y < 1$

(a)

$$\begin{aligned} & \int_0^2 \int_0^1 (x+2y)\,dy\,dx \\ = & \int_0^2 dx \int_0^1 (x+2y)\,dy \\ = & \int_0^2 dx \left[xy + y^2\right]_0^1 \\ = & \int_0^2 (x+1)\,dx \\ = & \frac{x^2}{2} + x\Big|_0^2 = 2 + 2 = 4. \end{aligned}$$

Therefore $C = \dfrac{1}{4}$.

(b) $f_X(x) = \displaystyle\int_0^1 f(x,y)\,dy = \frac{1}{4}\int_0^1 (x+2y)\,dy = \frac{x+1}{4}, \quad 0 < x < 2$

$f_Y(y) = \displaystyle\int_0^2 f(x,y)\,dx = \frac{1}{4}\int_0^2 (x+2y)\,dx = \frac{1}{2} + y, \quad 0 < y < 2$

X and Y are not independent since $f(x,y) \neq f_X(x)f_Y(y)$.

(c)

$$z = \frac{9}{(x+1)^2}$$

$x = 0 \Rightarrow z = 9$ and $x = 2 \Rightarrow z = 1$

$$\frac{dz}{dx} = -\frac{9}{(x+1)^4} \cdot 2(x+1)$$

Therefore

$$\begin{aligned} g(z) &= \frac{x+1}{4} \cdot \left|\frac{dx}{dz}\right|, \quad 1 < z < 9 \\ &= \frac{x+1}{4} \cdot \frac{(x+1)^3}{18} \\ &= \frac{(x+1)^4}{72} \\ &= \frac{81}{z^2 \cdot 72} = \frac{9}{8} \cdot \frac{1}{z^2}, \quad 1 < z < 9 \end{aligned}$$