Question
Suppose that the joint pdf of X and Y is given by
$f(x,y) = 2x \quad 0 \le x \le 1, \quad 0 \le y \le 1.$

(a) Find the conditional distribution of Y given that $X = x$,

(b) Find $P(Y \le \frac{1}{2} | X = 0.5)$ and $P(Y \le \frac{1}{2} | X = 0.75)$.

(c) Explain your answers from part b.

Answer
$f(x,y) = 2x, \quad 0 \le x \le 1; \quad 0 \le y \le 1$

(a) $f_X(x) = \int_0^1 2x\,dy = 2x \int_0^1 dy = 2x, \quad 0 \le x \le 1$

$f_Y(y) = \int_0^1 2x\,dx = x^2\Big|_0^1 = 1, \quad 0 \le y \le 1$

Therefore $f(y|x) = \dfrac{f(x,y)}{f_X(x)} = \dfrac{2x}{2x} = 1, \quad 0 \le y \le 1$

(b) $P(Y \le \frac{1}{2} | X = 0.5) = \int_0^{\frac{1}{2}} dy = \dfrac{1}{2} = \int_0^{\frac{1}{2}} f(y|x)\,dy$

$P(Y \le \frac{1}{2} | X = 0.75) = \int_0^{\frac{1}{2}} f(y|X = x)\,dy = \int_0^{\frac{1}{2}} dy = \dfrac{1}{2}$

(c) The two probabilities are equal because the distribution of $Y|X = x$ does not depend on x since X and Y are independent.