Question
A pdf is defined by

$$f(x,y) = \begin{cases} \frac{1}{4}(x+2y) & \text{if } 0 < x < 2 \text{ and } 0 < y < 1 \\ 0 & \text{otherwise} \end{cases}$$

(a) Find the correlation coefficient between X and Y.

(b) Determine the value of $\text{var}(2X - 3Y + 8)$.

Answer

$$\begin{aligned} E(XY) &= \frac{1}{4}\int_0^2\int_0^1 xy(x+2y)\,dy\,dx \\ &= \frac{1}{4}\int_0^2 x\,dx\int_0^1 (xy+2y^2)\,dy \\ &= \frac{1}{4}\int_0^2 \left(\frac{x^2}{2}+\frac{2x}{3}\right)\,dx \\ &= \frac{2}{3} \end{aligned}$$

$$\begin{aligned} E(X^2) &= \int_0^2 x^2\frac{x+1}{4}\,dx = \frac{1}{4}\left(\frac{x^4}{4}+\frac{x^3}{3}\right)\Big|_0^2 \\ &= \frac{1}{4}\left(4+\frac{8}{3}\right) = \frac{1}{4}\cdot\frac{20}{3} = \frac{5}{3} \end{aligned}$$

$$\begin{aligned} E(Y^2) &= \int_0^1 y^2\left(\frac{1}{2}+y\right)\,dy = \frac{1}{2}\left(\frac{y^3}{3}+\frac{y^4}{4}\right)\Big|_0^1 \\ &= \frac{1}{6}+\frac{1}{4} = \frac{10}{24} = \frac{5}{12} \end{aligned}$$

$$E(X) = \int_0^2 x\frac{x+1}{4}\,dx = \frac{1}{4}\left(\frac{x^3}{3}+\frac{x^2}{2}\right)\Big|_0^2 = \frac{1}{4}\left(\frac{8}{3}+2\right) = \frac{1}{4}\cdot\frac{14}{3} = \frac{7}{6}$$

$$E(Y) = \int_0^1 y\left(\frac{1}{2}+y\right)\,dy = \left(\frac{y^2}{4}+\frac{y^3}{3}\right)\Big|_0^1 = \frac{1}{4}+\frac{1}{3} = \frac{7}{12}$$

Therefore $\text{var}(X) = \frac{5}{3} - \left(\frac{7}{6}\right)^2 = \frac{11}{36}$ $\quad \text{var}(Y) = \frac{5}{12} - \left(\frac{7}{12}\right)^2 = \frac{11}{144}$

Therefore $\rho_{XY} = \frac{\frac{2}{3} - \frac{7}{6} \cdot \frac{7}{12}}{\sqrt{\frac{11}{36} \cdot \frac{11}{144}}} = -\frac{1}{11}$
$\text{var}(aX + bY + c) = a^2\text{var}(X) + b^2\text{var}(Y) + 2ab\text{cov}(X, Y)$
Therefore

$$\begin{aligned} \text{var}(2X - 3Y + 8) &= 4 \cdot \frac{11}{36} + (-3)^2 \frac{11}{144} + 2(2)(-3)(-\frac{1}{72}) \\ &= \frac{11}{9} + \frac{11}{6} + \frac{1}{6} = 2.076 \end{aligned}$$