Question

Suppose that X is uniform in the interval $(0, 2\pi)$ and Y, independent of X, is exponential with parameter 1.

(a) Find the joint pdf of U and V defined by

$$U = \sqrt{2Y}\cos(X),\ V = \sqrt{2Y}\sin(X).$$

(b) Show that U and V are independent, each having a standard normal distribution.

(c) State the distribution of $\tan(X)$.

Answer

$f(x, y) = \dfrac{1}{2\pi}e^{-y}, \quad x \in (0, 2\pi),\ y > 0$

Therefore $\begin{array}{rcl} x &=& \tan^{-1}\left(\frac{v}{u}\right) \\ y &=& \frac{u^2+v^2}{2} \end{array}$ Also $\begin{array}{rcl} -\infty &<& u < \infty \\ -\infty &<& v < \infty \end{array}$

$$J = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} \frac{1}{1+\frac{v^2}{u^2}}\frac{-v}{u^2} & \frac{1}{1+\frac{v^2}{u^2}}\frac{1}{u} \\ u & v \end{vmatrix} = -1$$

Therefore

$$\begin{aligned} f(u, v) &= \frac{1}{2\pi}e^{-\frac{1}{2}(u^2+v^2)} \cdot |-1|,\ -\infty < u < \infty,\ -\infty < v < \infty \\ &= \frac{1}{2\pi}e^{-\frac{1}{2}(u^2+v^2)} \\ &= \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}u^2} \cdot \frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}v^2} \end{aligned}$$

Therefore U and V are independent standard normals.

$\tan(X) = \dfrac{V}{U}$ where U and V are *indep* $N(0, 1)$.

Therefore $\tan(X) \sim$ Cauchy distribution.