Question

a) Find the Mobius transformation

$$w = \frac{az+b}{cz+d}$$

which maps the disc $|z| < 1$ onto the disc $|w-1| < 1$, and which maps the points $z = 0,\ z = 1$ onto the points $w = \frac{1}{2},\ w = 0$ respectively.

b) Find the real and imaginary parts of the function $\cos z$, where $z = x+iy$.

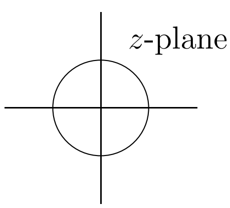
Find the images of the lines x =constant under the transformation $w = \cos z$, and use this to show that the transformation maps the infinite strip

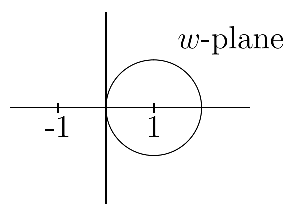
$$0 < x < \pi, \qquad y > 0$$

onto the lower half-plane im$w < 0$.

Answer

a)





The transformation must map the boundaries of the discs onto one another. Inverse pairs map to inverse pairs of points, so

$z = 0 \rightarrow w = \frac{1}{2} \Rightarrow z = \infty \rightarrow w = -1$

Thus $\frac{b}{d} = \frac{1}{2}$ and $\frac{a}{c} = -1$.

Now $z = 1 \rightarrow w = 0 \Rightarrow a + b = 0$

So we have $b = -a,\ c = -a,\ d = -2a$.

Hence $w = \dfrac{z-1}{-z-2}$

b) $\cos(x+iy) = \cos x \cos iy - \sin x \sin iy = \cos x \cosh y - i \sin x \sinh y$

So $u = \cos x \cosh y$ and $v = -\sin x \sinh y$

For each x=constant this gives parametric equations for hyperbolae. Now consider $0 < x < \pi$.

For $x = \frac{\pi}{2}$ $\cos x = 0$ $\sin x = 1$

So $u = 0$, $v = -\sinh y < 0$ for $y > 0$

So $x = \frac{\pi}{2}$ $y < 0$ maps to the negative imaginary axis $u = 0$, $v < 0$.

For $0 < x < \frac{\pi}{2}$ $\cos x > 0$ and $\sin x > 0$.

For $y > 0$ $\cosh y > 0$ and $\sinh y > 0$.

So lines x=constant and $y > 0$ in this range map to hperbolas in the fourth quadrant.

Similarly for $\frac{\pi}{2} < x < \pi$ the images are parts of hyperbolas in the third quadrant.

DIAGRAMS