Question

State Rouche's Theorem, and use it to show that all the roots of the equation

$$z^6 + (1+i)z + 1 = 0$$

lie in the annulus $\frac{1}{2} \le |z| < \frac{5}{4}$.

Use the argument principle to determine how many of these roots lie in the quadrant $0 < \arg z < \frac{1}{2}\pi$.

Answer

Rouche's Theorem states that if $f(z)$ and $g(z)$ are both analytic inside and on the closed contour C, and if $|g(z)| < |f(z)|$ on C then $f(z)$ and $F(z)+g(z)$ have the same number of zeros inside C.

i) Let $f(z) = 1$, $g(z) = z^6 + (1+i)z$.

   Then for $|z| = \frac{1}{2}$, $|g(z)| \le (\frac{1}{2})^6 + \frac{1}{2}\sqrt{2} < 1 = |f(z)|$

   $f(z)$ has no zeros inside $|z| = \frac{1}{2}$, and so

   $f(z) + g(z)$ has none inside $|z| = \frac{1}{2}$.

ii) Let $f(z) = z^6$, $g(z) = (1+i)z + 1$

   Then for $|z| = \frac{5}{4}$, $|g(z)| \le \sqrt{2}\frac{5}{4} + 1 \approx 2.77$

   $|f(z)| = (\frac{5}{4})^6 \approx 3.81$

   $f(z)$ has six zeros inside $|z| = \frac{5}{4}$, and so

   $f(z) + g(z)$ has all six inside $|z| = \frac{5}{4}$.

Now consider the contour C in the first quadrant.

DIAGRAM

I. On OA $f = x^6 + x + 1 + ix$ and $\tan \arg z = \dfrac{x}{x^6 + x + 1}$. This is continuous for $x > 0$, it is zero at 0 and tends to zero as $R \to \infty$. So $[\arg f(z)]_{OA} = \epsilon$ (something small)

II. On BO $z = iy$ so $f = -y^6 - y + 1 + iy$ and $\tan \arg z = \dfrac{y}{1 - y - y^6}$. Now the derivative of $1 - y - y^6$ is $-1 - 6y^5$ which is negative for all $y > 0$. So $1 - y - y^6$ has just one positive root. Thus the graph of $\tan \arg z$ is

   DIAGRAM

   Hence $[\arg z]_{BO} = -\pi + \delta$ (δ is small)

III. On AB $z = Re^{i\theta}$ and $f(z) = R^6 e^{6i\theta}(1 + w)$, $|w|$ is small.

So as θ goes from 0 to $\pi 2$, $[\arg f(z)]_{AB} = 3\pi + \eta$, η is small.

Thus $\frac{1}{2\pi}[\arg f(z)]_C = 1$ since it must be an integer.
Thus the equation has 1 root in the first quadrant.