Question

a) Show that

$$\int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \frac{2\pi}{\sqrt{(a^2 - b^2)}},$$

where $a > b > 0$.

b) Evaluate the sum of the series

$$\sum_{n=1}^{\infty} \frac{1}{n^2 - a^2},$$

where a is not an integer. Give detailed justification of your method.

Answer

a) Let C be the unit circle, $z = e^{i\theta}$, $d\theta = \frac{dz}{iz}$, $\cos\theta = \frac{1}{2}(z + \frac{1}{z})$. Thus

$$I = \int_0^{2\pi} \frac{d\theta}{a + b\cos\theta} = \int_C \frac{dz}{iz}\left(a + \frac{b}{2}\left(z + \frac{1}{z}\right)\right) = \frac{2}{ib}\int \frac{dz}{z^2 + \frac{2a}{b}z + 1}$$

The integrand has simple poles at

$$z = -\frac{a}{b} + \sqrt{\frac{a^2}{b^2} - 1} = \alpha_1 \text{ inside } C$$

$$z = -\frac{a}{b} - \sqrt{\frac{a^2}{b^2} - 1} = \alpha_2 \text{ outside } C$$

The residue at $z = \alpha_1$ is $\dfrac{1}{\alpha_1 - \alpha_2} = \dfrac{b}{2\sqrt{a^2 - b^2}}$

So $I = \dfrac{2}{ib}2\pi i.\text{res}(\alpha_1) = \dfrac{2\pi}{\sqrt{(a^2 - b^2)}}$

b) The function $f(z) = \dfrac{\pi\cot\pi z}{z^2 - a^2}$ has a simple pole at $z = n$ with residue $\dfrac{1}{n^2 - a^2}$ for $n \in \mathbf{Z}$, and simple poles at $z = \pm a$ with residue $\dfrac{\pi\cot\pi a}{2a}$ at each.

Now let C_N be the square with vertices $\pm(N+\frac{1}{2})(1\pm i) \quad N \in \mathbf{N}$

On the upper edge $z = x + (N+\frac{1}{2})i$, and so

$$|\cot \pi z| = \left|\frac{\cos \pi z}{\sin \pi z}\right| = \left|i\frac{e^{i\pi z}+e^{-i\pi z}}{e^{i\pi z}-e^{-i\pi z}}\right| = \left|\frac{e^{2i\pi z}+1}{e^{2i\pi z}-1}\right|$$

$$\leq \frac{1+|e^{2\pi i z}|}{1-|e^{2\pi i z}|} = \frac{1+e^{-2\pi(N+\frac{1}{2})}}{1-e^{-2\pi(N+\frac{1}{2})}} \leq \frac{2}{1-e^{-\pi}} \text{ for all } N \in \mathbf{N}.$$

Since $|\cot \pi z| = |\cot \pi(-z)|$ the same bound serves on the lower side of the square.

On the sides parallel to the imaginary axis

$$|\cot \pi z| = |\cot \pi(\pm N + \tfrac{1}{2} + iy)|$$

$$= |\cot \pi(\tfrac{1}{2} + iy)| = |-\tan \pi i y| = |\tanh y| \leq 1$$

So $\exists K\ \forall N \in \mathbf{N},\ \forall z \in C_N,\ |\pi \cot \pi z| \leq K$

Now provided $N \geq |a|$, we have

$$\int_{C_N} f(z)dz = 2\pi i \left\{ \sum_{n=-N}^{N} \frac{1}{n^2-a^2} + \frac{2\pi \cot \pi a}{2a} \right\}$$

$$\text{Now } \left|\int_{C_N} \frac{\pi \cot \pi z}{z^2-a^2} dz\right| \leq \frac{K8\left(N+\frac{1}{2}\right)}{\left(N+\frac{1}{2}\right)^2 - |a|^2} \to 0 \text{ as } N \to \infty.$$

since $|z| \geq N + \frac{1}{2}$ on C_N.

Letting $N \to \infty$ therefore gives

$$\sum_{n=-\infty}^{\infty} \frac{1}{n^2-a^2} = -\frac{\pi \cot \pi a}{a}$$

$$\text{Thus } \sum_{n=1}^{\infty} \frac{1}{n^2-a^2} = \frac{1}{2a^2} - \frac{\pi \cot \pi a}{2a}$$