Question

a) Find all values of 4^{4i}, and show that they form a set of point lying on a straight line in the complex plane. Find the limit point of this set.

b) Let C denote the semi-circle

$$\left\{e^{it} : -\frac{1}{2}\pi \leq t \leq \frac{1}{2}\pi\right\}.$$

Evaluate the contour integral

$$\int_C \text{Log} z dz,$$

where Log$z$ denotes the principle logarithm of z.

c) Let K denote the circle, centre $1+i$ and radius 1. Evaluate the contour integral

$$\int_K \frac{\text{Log} z}{z-(1+i)} dz,$$

giving your answer in the form $a+ib$, where a and b are real.

Answer

a) $a^b = \exp(b \log a)$

So $4^{4i} = \exp 4i(\ln 4 + 2n\pi i) \qquad n \in \mathbf{Z}$

$= \exp(i4\ln 4)\exp(-8n\pi)$

These points all lie on the line $\arg z = 4\ln 4$, and have modulus $\exp(-8n\pi)$. They therefore have limit point zero.

b) $\displaystyle\int_C \text{Log} z dz = \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} (\text{Log}|e^{it}| + i \arg e^{it}) ie^{it} dt$

$\displaystyle= \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} -te^{it} dt = \left[-t\frac{e^i t}{i}\right]_{\frac{\pi}{2}}^{\frac{\pi}{2}} + \int_{\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{e^i t}{i} dt$

$\displaystyle= 0 + \left[-e^{it}\right]_{\frac{\pi}{2}}^{\frac{\pi}{2}} = -2i$

c) Since K does not meet the negative real axis, or contain 0, $\mathrm{Log} z$ is analytic inside and on K. Thus

$$\int_K \frac{\mathrm{Log} z}{z-(1+i)} dz = 2\pi i \mathrm{Log}(1+i) = 2\pi i \left(\ln\sqrt{2} + i\frac{\pi}{4}\right) = -\frac{\pi^2}{2} + \pi i \ln 2$$